Finite groups with a metacyclic Frobenius group of automorphisms

Natalia Makarenko
(main results are joint with E. I. Khukhro)

Université de Haute Alsace, Sobolev Institute of Mathematics, Novosibirsk

Groups St Andrews 2013

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- H acts faithfully on every non-trivial H-invariant subgroup of F. It follows that all abelian subgroups of H are cyclic.
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- By Thompson's theorem (1959) a finite group admitting a fixed-point-free automorphism of prime order is nilpotent. Hence F is nilpotent.
- By Higman's theorem (1957) the nilpotency class of a finite (nilpotent) group admitting a fixed-point-free automorphism of prime order p is bounded in terms of p. Hence the nilpotency class of F is bounded in terms of the least prime divisor of $|H|$.

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- Part (a). Can the nilpotency class of G be bounded in terms of $|H|$ and the nilpotency class of $C_G(H)$?
- Part (b). Can the exponent of G be bounded in terms of $|H|$ and the exponent of $C_G(H)$?

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Let GFH be a double Frobenius group with complement H of order q. Assume that $C_G(H)$ is nilpotent of class c. Then G is nilpotent of (c, q)-bounded class.

Positive answer to the part a) of Mazurov's question.

Fixed-point-free action of Frobenius kernel

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Remark. By Higman-Kreknin-Kostrikin Theorem G is nilpotent of class at most $h(p)$, where $h(p)$ is Higman's function.

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Hypothesis. Principal properties (nilpotency class, exponent, order, rank) of G are completely defined by H and the action of H on G (and do not depend on F).

Order, rank, nilpotency

Theorem (Khukhro–Makarenko–Shumyatsky, 2010, to appear in Forum Mathematicum)

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The proof is based on the fact that all FH-invariant elementary abelian sections of G are free $\mathbb{F}_p H$-modules (for various p) by Clifford's theorem.

Definition. Recall that for a group A and a field k, a free kA-module of rank n is a direct sum of n copies of the group algebra kA each of which is regarded as a vector space over k of dimension $|A|$ with a basis $\{v_g, g \in A\}$ labelled by elements of A on which A acts in a regular permutation representation: $v_g h = v_{gh}$.

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Let a Lie ring L admit a Frobenius group of automorphisms FH with cyclic kernel F of finite order n and complement H of order q.

If $C_L(F) = 0$ and $C_L(H)$ is nilpotent of class c, then the nilpotency class of L is bounded in terms of q and c.

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Note, that by Kreknin's theorem L is solvable of n-bounded derived length.

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Khukhro (2013, Algebra and Logic): Restrictions for the order and rank of G.

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Main Theorem (Khukhro, Makarenko, J. of Algebra 2013).

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Then G has a nilpotent characteristic subgroup of index bounded in terms of c, $|C_G(F)|$, and $|FH|$ whose nilpotency class is bounded in terms of c and $|H|$ only.

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Proof.

- Reduction to soluble groups is given by results based on the classification of Hartley (1992) and Wang–Cheng (1993).
- Reduction to the case of a nilpotent group G by bounding the index of the Fitting subgroup by representation theory arguments.

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Theorem (Khukhro, Makarenko, J. of Algebra 2013).

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Then the index of the Fitting subgroup $F(G)$ is bounded in terms of $|C_G(F)|$ and $|F|$.

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Direct application of Lie ring Theorem to the associated Lie ring $L(G)$ does not give a required result for the group G, since there is no good correspondence between subgroups of G and subrings of $L(G)$.

We overcome this difficulty by proving that, in a certain critical situation, the group G itself is nilpotent of (c, q)-bounded class, the advantage being that the nilpotency class of G is equal to that of $L(G)$. This is not true in general, but can be achieved by using induction on a certain complex parameter.

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Suppose that a finite Frobenius group FH with cyclic kernel F and complement H acts by automorphisms on a Lie ring L in whose ground ring $|F|$ is invertible. Let the fixed-point subring $C_L(H)$ of the complement be nilpotent of class c and the fixed-point subring of the kernel $C_L(F)$ be finite of order m.

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Then L has a nilpotent Lie subring whose index in the additive group of L is bounded in terms of c, m, and $|F|$ and whose nilpotency class is bounded in terms of c and $|H|$ only.

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- the K–M–S theorem on Lie rings with fixed-point-free action of Frobenius kernel F, reformulated as a certain combinatorial fact about Lie rings with finite cyclic grading.
- Method of graded centralizers. (This method was created by Khukhro for almost regular automorphisms of prime order and later developed in further studies of almost fixed-point-free automorphisms of Lie rings and nilpotent groups.)

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Open problem. Let a finite Frobenius group FH with cyclic kernel F and complement H act by automorphisms on a finite group G. Suppose that the fixed-point subgroup $C_G(H)$ is soluble of derived length c and $C_G(F) = 1$. Is the derived length of G bounded in terms of that of $C_G(H)$ and $|H|$?

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The last problem is already reduced to nilpotent groups, since it was proved by Khukhro that if $C_G(F) = 1$, then the Fitting height of G is equal to that of $C_G(H)$.

Summary table

G	FH is a Frobenius group; $C_G(F)=1$	FH is a Frobenius group; $\lvert C_G(F)\rvert = m$	$\lvert\varphi\rvert = p$ is prime; $C_G(\varphi)=1$	$\lvert\varphi\rvert$ is not prime; $C_G(\varphi)=1$
	Kh-M-Sh, 2010:	Khukhro, 2013:		
$\lvert G\rvert$	$\lvert G\rvert = \lvert C_G(H)\rvert^{\lvert H\rvert}$;	restrictions on the order		
rk G	$\mathrm{rk}\, G \leq f(\mathrm{rk}\, H, \lvert H\rvert)$;	and the rank.		
nilpotency	if H is nilpotent $\Rightarrow$ G is nilpotent;	M-Kh, 2013: $F(G)$ has index bounded in terms in $\lvert F\rvert$ and $\lvert C_G(F)\rvert$;	Tompson, 1959: G is nilpotent.	There is a reduction to nilpotent groups.
$\mathrm{cl}\,(G)$	If F is cyclic and $C_G(H)$ is nilpotent of class c $\Rightarrow$ the nilpotency class of G is bounded in terms of c and $\lvert H\rvert$.	If F is cyclic and $C_G(H)$ is nilpotent of class c $\Rightarrow$ there is a characteristic subgroup G_1 such that $\mathrm{cl}\,(G_1) \leq f(c, \lvert H\rvert)$, $\lvert G : G_1\rvert \leq g(\lvert FH\rvert, \lvert C_G(F)\rvert, c)$.	Higman–Kreknin–Kostrikin, 1957, 1963: G is nilpotent of class bounded in terms of p.	????