# On the influence of subgroups on the structure of finite groups

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**Theorem (R. Dedekind, 1896)**

*A group $G$ is Dedekind if and only if $G$ is abelian or $G$ is a direct product of the quaternion group $Q_8$ of order 8, an elementary abelian 2-group and an abelian group of odd order.*

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**Theorem (K. Iwasawa, 1941)**

*Let $p$ be a prime. A $p$-group $G$ is an Iwasawa group if and only if $G$ is a Dedekind group, or $G$ contains an abelian normal subgroup $N$ such that $G/N$ is cyclic and so $G = \langle x \rangle N$ for an element $x$ of $G$ and $a^x = a^{1+p^s}$ for all $a \in N$, where $s \geqslant 1$ and $s \geqslant 2$ if $p = 2$.*

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**Examples of $T$-groups:**

- Dedekind groups = nilpotent $T$-groups;
- simple groups.

**Theorem (W. Gaschütz, 1957)**

*A group $G$ is a soluble $T$-group if and only if the following conditions are satisfied:*

1. *the nilpotent residual $L$ of $G$ is an abelian Hall subgroup of odd order;*
2. *$G$ acts by conjugation on $L$ as a group of power automorphisms, and*
3. *$G/L$ is a Dedekind group.*

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**Definition**

A group $G$ is said to be a *PT-group* when
if $H$ is a permutable subgroup of $K$ and $K$ is a permutable subgroup of $G$, then $H$ is a permutable subgroup of $G$.

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A group $G$ is a *PST-group* when
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The $PST$-groups are exactly the groups in which every subnormal subgroup is $s$-permutable.

**Theorem (R.K. Agrawal, 1975)**

*Let $G$ be a group with nilpotent residual $L$. The following statements are equivalent:*

1. *$L$ is an abelian Hall subgroup of odd order in which $G$ acts by conjugation as a group of power automorphisms;*
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2. *$G$ is a soluble $PST$-group.*

**Corollary**

*Let $G$ be a group.*

1. *$G$ is a soluble $PT$-group if and only if $G$ is a soluble $PST$-group whose Sylow subgroups are Iwasawa groups;*
2. *$G$ is a soluble $T$-group if and only if $G$ is a soluble $PST$-group whose Sylow subgroups are Dedekind groups.*

$$
\begin{array}{ccccc}
T & \subsetneq & PT & \subsetneq & PST \\
\rotatebox{90}{$\subsetneq$} & & \rotatebox{90}{$\subsetneq$} & & \rotatebox{90}{$\subsetneq$} \\
\text{Dedekind} & \subsetneq & \text{Iwasawa} & \subsetneq & \text{nilpotent}
\end{array}
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In the soluble universe:

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\begin{array}{ccccccc}
T & \subsetneq & PT & \subsetneq & PST & \subsetneq & \text{supersoluble} \\
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If $H$ is a subgroup of a group $G$, we denote by $H^G$ the *normal closure of $H$ in $G$*, that is, the smallest normal subgroup of $G$ containing $H$.

**Theorem (Y. Li, 2006)**

*Let $G$ be a group. The following statements are equivalent:*

1. *$G$ is a soluble $T$-group;*
2. *$N_G(H) \cap H^G = H$ for all subgroups $H$ of $G$;*
3. *$N_G(H) \cap H^G = H$ for all $p$-subgroups $H$ of $G$ and every prime $p$.*

**Theorem (O.H. Kegel, 1962)**

*If $H_1$ and $H_2$ are two s-permutable subgroups of the group $G$, then $H_1 \cap H_2$ is an s-permutable subgroup of $G$. Consequently, the set of all s-permutable subgroups is a sublattice of the subnormal subgroup lattice.*

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**Definition**

Let $H$ be a subgroup of a group $G$.

1. The *permutable closure $A_G(H)$ of $H$ in $G$* is the intersection of all permutable subgroups of $G$ containing $H$.
2. The *s-permutable closure $B_G(H)$ of $H$ in $G$* is the intersection of all $s$-permutable subgroups of $G$ containing $H$.

## Example

Assume that $p$ is an odd prime,

$$A = \langle a, b \mid a^{p^2} = b^p = 1,\ a^b = a^{1+p} \rangle$$

is an extraspecial group of order $p^3$ and exponent $p^2$ and $Z = \langle z \rangle$ is a cyclic group of order $p^2$. Consider $G = A \times Z$. Then $A \lhd G$ and $B = \langle b \rangle \times \langle z \rangle$ is permutable in $G$ since $\langle b \rangle$ is permutable in $A$. But $A \cap B = \langle b \rangle$ is not permutable in $G$, since $\langle b \rangle$ does not permute with $\langle az \rangle$. For a subgroup $H = \langle b \rangle$, the permutable closure $A_G(H) = H$ is not permutable in $G$.

**Theorem (A. Ballester-Bolinches, R. Esteban-Romero, Y. Li, 2010)**

*Let $G$ be a group. The following statements are equivalent:*

1. *$G$ is a soluble $PT$-group;*
2. *$N_G(H) \cap A_G(H) = H$ for every subgroup $H$ of $G$;*
3. *$N_G(H) \cap A_G(H) = H$ for every $p$-subgroup $H$ of $G$ and every prime $p$.*

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*Let $G$ be a group. The following statements are equivalent:*

1. *$G$ is a soluble $PST$-group;*
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A subgroup $H$ of a group $G$ is said to be a *PR-subgroup* of $G$ (Permutable Restriction) if, whenever $K \trianglelefteq H$, $A_G(K) \cap H = K$.

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**Example**

Let $G = A_5$, the alternating group of degree 5. Then every 5-subgroup of $G$ is an $NR$-subgroup of $G$, a $PR$-subgroup of $G$ and an $sPR$-subgroup of $G$. Let $H = \langle(12345)\rangle$. Hence $|N_G(H)| = 10$ and $H^G \cap N_G(H) = N_G(H) \cap A_G(H) = N_G(H) \cap B_G(H) = N_G(H) \neq H$.

## Example

Let $G$ be the semidirect product of a quaternion group $P$ of order 8 with a cyclic group $Q$ of order 3, which induces an automorphism permuting cyclically the three maximal subgroups of the quaternion group. Then every 3-subgroup of $G$ is an $NR$-subgroup of $G$, a $PR$-subgroup of $G$ and an $sPR$-subgroup of $G$. But

$$\begin{aligned} Q^G \cap N_G(Q) = A_G(Q) \cap N_G(Q) = B_G(Q) \cap N_G(Q) \\ = G \cap QP' = QP' \neq Q. \end{aligned}$$

**Theorem (I.A.M., 2012)**

*Let $G$ be a group. The following statements are equivalent:*

1. *$G$ is a soluble $T$-group;*
2. *each subgroup of $G$ is an NR-subgroup of $G$;*
3. *for each prime $p \in \pi(G)$, each $p$-subgroup of $G$ is an NR-subgroup of $G$.*

A subgroup $H$ of a group $G$ is *normal sensitive in $G$* if the following holds:

$$\{N \mid N \text{ is normal in } H\} = \{H \cap W \mid W \text{ is normal in } G\}.$$

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**Corollary (I.A.M., 2012)**

*A group $G$ is a soluble $T$-group if and only if for every $p \in \pi(G)$, every $p$-subgroup of $G$ is normal sensitive in $G$.*

**Theorem (I.A.M., 2012)**

*Let $G$ be a group. The following statements are equivalent:*

1. *$G$ is a soluble $PT$-group;*
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A subgroup $H$ of a group $G$ is *permutable sensitive in $G$* if the following holds:

$$\{N \mid N \text{ is permutable in } H\} = \{H \cap W \mid W \text{ is permutable in } G\}.$$

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**Theorem (J.C. Beidleman, M.F. Rangland, 2007)**

*Every subgroup of a group $G$ is permutable sensitive in $G$ if and only if $G$ is a soluble $PT$-group.*

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*A group $G$ is a soluble $PT$-group if and only if for every $p \in \pi(G)$, every $p$-subgroup of $G$ is permutable sensitive in $G$.*

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## Example

Let $P = \langle a, b \mid a^3 = b^{3^2} = 1, b^a = b^4 \rangle$ be a metacyclic group of order $3^3$ and exponent $3^2$. Let $x$ be the automorphism of $P$ of order 2 given by $a^x = a$, $b^x = b^{-1}$. Let $H = P \rtimes \langle x \rangle$ be the corresponding semidirect product and let $G = H \times C$, where $C = \langle c \rangle$ is cyclic of order 3. Then a subgroup $\langle a, bc \rangle$ is a $PR$-subgroup of $G$. But it is not permutable sensitive in $G$.

**Theorem (I.A.M., 2012)**

*Let $G$ be a group. The following statements are equivalent:*

1. *$G$ is a soluble $PST$-group;*
2. *each subgroup of $G$ is an sPR-subgroup of $G$;*
3. *for each prime $p \in \pi(G)$, every $p$-subgroup of $G$ is an sPR-subgroup of $G$.*

**Theorem (I.A.M., 2012)**

*Let $G$ be a group. The following statements are equivalent:*

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3. *for each prime $p \in \pi(G)$, every $p$-subgroup of $G$ is an sPR-subgroup of $G$.*

A subgroup $H$ of a group $G$ is *s-permutable sensitive in $G$* if the following holds:

$$\{N \mid N \text{ is } s\text{-permutable in } H\} = \{H \cap W \mid W \text{ is } s\text{-permutable in } G\}.$$

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**Theorem (J.C. Beidleman, M.F. Rangland, 2007)**

*Every subgroup of a group $G$ is s-permutable sensitive in $G$ if and only if $G$ is a soluble $PST$-group.*

**Corollary (I.A.M., 2012)**

*A group $G$ is a soluble $PST$-group if and only if for every $p \in \pi(G)$, every $p$-subgroup of $G$ is s-permutable sensitive in $G$.*

**Corollary (I.A.M., 2012)**

*A group $G$ is a soluble $PST$-group if and only if for every $p \in \pi(G)$, every $p$-subgroup of $G$ is $s$-permutable sensitive in $G$.*

**Example**

Let $G$ be the direct product of a symetric group of degree 4 and a cyclic group of order 2. Let $H = \langle (1,2), (1,3)(2,4)(5,6), (1,2)(3,4) \rangle$ (here $(5,6)$ generates the cyclic subgroup of order 2). Then $H$ is an $sPR$-subgroup of $G$, but it is not $s$-permutable sensitive in $G$.

A subgroup $H$ of $G$ is an *$\mathcal{H}$-subgroup of $G$* if $N_G(H) \cap H^g \leqslant H$ for all $g \in G$.

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Let $p$ be a prime. A group $G$ satisfies:

- *the property $NR_p$* if a Sylow $p$-subgroup of $G$ is an $NR$-subgroup of $G$;
- *the property $\mathcal{H}_p$* if every maximal subgroup of a Sylow $p$-subgroup of $G$ is an $\mathcal{H}$-subgroup of $G$.

A subgroup $H$ of $G$ is an *$\mathcal{H}$-subgroup of $G$* if $N_G(H) \cap H^g \leqslant H$ for all $g \in G$.

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**Theorem (I.A.M., 2012)**

*Let $G$ be a group. The following conditions are equivalent:*

1. *$G$ is a soluble $PST$-group;*
2. *every subgroup of $G$ satisfies $NR_p$ for every prime $p \in \pi(G)$;*
3. *every subgroup of $G$ satisfies $\mathcal{H}_p$ for every prime $p \in \pi(G)$.*

**Theorem (I.A.M., 2012)**

*Let $G$ be a group all of whose second maximal subgroups of even order are soluble PST-groups. Then $G$ is either a soluble group or one of the following groups:*

1. $PSL(2, 2^f)$, *where $f$ is a prime such that $2^f - 1$ is a prime;*
2. $PSL(2, p)$, *where $p$ is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod 5$ and $p \equiv 3$ or $5 \pmod 8$;*
3. $PSL(2, 3^f)$, *where $f$ is an odd prime and $3^f \equiv 3 \pmod 8$;*
4. $SL(2, 3^f)$, *where $f$ is an odd prime, $3^f \equiv 3 \pmod 8$ and $(3^f - 1)/2$ is a prime;*
5. $SL(2, p)$, *where $p$ is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod 5$ and $p \equiv 3$ or $5 \pmod 8$;*

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2. $PSL(2, p)$, *where $p$ is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod 5$ and $p \equiv 3$ or $5 \pmod 8$;*
3. $PSL(2, 3^f)$, *where $f$ is an odd prime, $3^f \equiv 3 \pmod 8$ and $(3^f - 1)/2$ is a prime;*
4. $SL(2, p)$, *where $p$ is a prime with $p > 3$, $p^2 - 1 \not\equiv 0 \pmod 5$ and $p \equiv 3$ or $5 \pmod 8$.*

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$G$ satisfies $\mathcal{X}_p$ if every subgroup of a Sylow $p$-subgroup $P$ of $G$ is permutable in $N_G(P)$,

$G$ satisfies $\mathcal{Y}_p$ if, whenever $H$ and $K$ are $p$-subgroups of $G$ with $H \leqslant K$, $H$ is $s$-permutable in $N_G(K)$.

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$G$ satisfies $\mathcal{Y}_p$ if, whenever $H$ and $K$ are $p$-subgroups of $G$ with $H \leqslant K$, $H$ is $s$-permutable in $N_G(K)$.

**Theorem (A. Ballester-Bolinches, R. Esteban-Rormero, 2002)**

*A group $G$ satisfies $\mathcal{X}_p$ (respectively, $\mathcal{C}_p$) if and only if $G$ satisfies $\mathcal{Y}_p$ and the Sylow $p$-subgroups of $G$ are Iwasawa (respectively, Dedekind).*

Let $p$ be a prime.

$$\begin{array}{ccccc}
\mathcal{C}_p & \subsetneq & \mathcal{X}_p & \subsetneq & \mathcal{Y}_p \\
\cup\!\nparallel & & \cup\!\nparallel & & \cup\!\nparallel \\
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**Theorem (D.J.S. Robinson, 1968)**

*A group $G$ is a soluble $T$-group if and only if $G$ satisfies $\mathcal{C}_p$ for all $p \in \pi(G)$.*

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**Theorem (J.C. Beidleman, B. Brewster, D.J.S. Robinson, 1999)**

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A subgroup $H$ of a group $G$ is said to be *pronormal in $G$* if for every $g \in G$, $H$ and $H^g$ are conjugate in their join $\langle H, H^g \rangle$.

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**Theorem**

*Let $G$ be a group and let $p$ be a prime. Then:*

1. (A. Ballester-Bolinches, R. Esteban-Romero, M. Asaad, 2010) *$G$ satisfies $\mathcal{Y}_p$ if and only if every subgroup of $G$ satisfies $\mathbf{H}_p$;*
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**Proposition**

*Let $G$ be a group and let $p$ be a prime.*

1. *If $G$ satisfies $NR_p$, then $G$ satisfies the property $\mathcal{H}_p$.*
2. *If $G$ satisfies $NR_p$, then $G$ satisfies the property $\mathbf{H}_p$.*

**Example**

Let $p$ be an odd prime and let $A = \langle a, b, c \mid a^p = b^p = c^p = 1, \ [a, b] = c \rangle$ be an extraspecial group of order $p^3$ and exponent $p$. Let $B = \langle x \rangle$ be a cyclic group of order $p$ and $P = A \times B$. Let $y$ be an automorphism of $P$ of order 2 given by $a^y = a^{-1}$, $b^y = b^{-1}$, $x^y = x^{-1}$. Let $G = P \rtimes \langle y \rangle$ be the corresponding semidirect product. Then every maximal subgroup of $P$ is normal in $G$, so is an $\mathcal{H}$-subgroup of $G$. Hence $G$ satisfies $\mathcal{H}_p$. But $H = \langle xc \rangle$ is normal in $P$, $\langle xc \rangle^y = \langle x^{-1}c \rangle$, $\langle xc \rangle$ and $\langle x^{-1}c \rangle$ are not conjugate in $\langle x, c \rangle$. Therefore $G$ satisfies neither $\mathbf{H}_p$ nor $NR_p$.

**Theorem (I.A.M., 2012)**

*Let $p$ be a prime and let $G$ be a $p$-soluble group. Then every subgroup of $G$ satisfies $\mathbf{H}_p$ if and only if every subgroup of $G$ satisfies $NR_p$.*

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**Example**

Let $G = PSL(2, 53)$. Since a Sylow 3-subgroup of $G$ is cyclic of order $3^3$, $G$ and its subgroups satisfy $\mathbf{H}_3$ and $\mathcal{H}_3$, but $G$ does not satisfy $NR_3$.

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### Question

Let $G$ be a non-$p$-soluble group. Is it true that every subgroup of $G$ satisfies $\mathbf{H}_p$ if and only if every subgroup of $G$ satisfies $\mathcal{H}_p$?

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Let $G = PSL(2, 53)$. Since a Sylow 3-subgroup of $G$ is cyclic of order $3^3$, $G$ and its subgroups satisfy $\mathbf{H}_3$ and $\mathcal{H}_3$, but $G$ does not satisfy $NR_3$.

**Question**

Let $G$ be a non-$p$-soluble group. Is it true that every subgroup of $G$ satisfies $\mathbf{H}_p$ if and only if every subgroup of $G$ satisfies $\mathcal{H}_p$ ?

**Question**

Assume that $G$ is a $p$-soluble group and $G$ has Iwasawa Sylow $p$-subgroups. Is it true that $G$ satisfies $NR_p$ if and only if $G$ satisfies $\mathcal{H}_p$ ?

**Corollary (I.A.M., 2012)**

*Let $p$ be a prime and let $G$ be a $p$-soluble group. Then:*

1. *$G$ satisfies $\mathcal{Y}_p$ if and only if every subgroup of $G$ satisfies $NR_p$.*
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**Theorem (I.A.M., 2012)**

*Let $G$ be a group. The following conditions are equivalent:*

1. *$G$ is a soluble $PT$-group;*
2. *$G$ satisfies $NR_p$ and $G$ has Iwasawa Sylow $p$-subgroups for all $p \in \pi(G)$.*

# References


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and references in them.

# Thank you