Zassenhaus Conjecture for cyclic-by-abelian groups

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Talk Structure

1. The Zassenhaus Conjectures and some results.
2. The cyclic-by-abelian case and the proof strategy.
3. Some methods used in the proof.

Notations and Setting

- G a finite group
- RG group ring of G over the ring R
- $U(RG)$ unit group of RG
- $\varepsilon : RG \to R$ augmentation map,

  $$\varepsilon(\sum_{g\in G} r_g g) = \sum_{g\in G} r_g$$
- $V(RG)$ the units of augmentation 1 aka normalized units
- If $R = \mathbb{Z}$, then $U(\mathbb{Z}G) = \pm V(\mathbb{Z}G)$, so suffices to study $V(\mathbb{Z}G)$

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Some basic results on torsion units

General results on torsion units ('40s-'60s):

- $u = \sum_{g \in G} z_g g \in V(\mathbb{Z}G)$ of finite order, then $z_1 \neq 0$ implies $u = 1$ (Berman-Higman).
- A a finite abelian group. Then $U(\mathbb{Z}A) = \pm A \times F$, where F is torsion-free abelian.
- $H \leq V(\mathbb{Z}G)$, H finite. Then $|H|$ divides $|G|$.
- $\exp(V(\mathbb{Z}G)) = \exp(G)$.

The Zassenhaus Conjectures and some results

Conjectures (H.J. Zassenhaus, in the '60s)

(ZC1): For $u \in V(\mathbb{Z}G)$ of finite order there exists $x \in U(\mathbb{Q}G)$ s.t. $x^{-1}ux \in G$.

(ZC2): For $H \leq V(\mathbb{Z}G)$ with $|H| = |G|$ there exists $x \in U(\mathbb{Q}G)$ s.t. $x^{-1}Hx = G$.

(ZC3): For $H \leq V(\mathbb{Z}G)$ with $|H|$ finite there exists $x \in U(\mathbb{Q}G)$ s.t. $x^{-1}Hx \leq G$.

A. Weiss gave a positive answer to (ZC3) for nilpotent groups in 1991. Roggenkamp and Scott gave a couterexample to (ZC2) and so also to (ZC3) in 1991. Hertweck and Blanchard even found a couterxample to (ZC2) of order 96. The only conjecture left open is (ZC1) and for this reason it is called the Zassenhaus Conjecture.

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Results on (ZC1)

There has been a lot of work on the Zassenhaus Conjecture, namely it is known to hold for

- Nilpotent groups (Weiss '91)
- Groups having a normal Sylow subgroup with an abelian complement (Hertweck '06)
- Cyclic-by-abelian groups ($\geq$ 8 papers '83-'06; Hertweck '08 and Caicedo, del Río, M. '12)
- Some families of metabelian groups, not necessarily cyclic-by-abelian (Sehgal, Weiss '86; Marciniak, Ritter, Sehgal, Weiss '87)
- Some concrete groups (see A. Bächles talk)

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The Cyclic-By-Abelian Case

Theorem (Hertweck '08)

Let $G = AX$ with $A \trianglelefteq G$ cyclic and X abelian. Then (ZC1) holds for G.

This proved especially the metacyclic case, which had been open till then.

Relying on this results and developing the methods further we got:

Theorem (Caicedo, M, del Río '12)

Let G be cyclic-by-abelian, i.e. G has a cyclic normal subgroup A s.t. G/A is abelian. Then (ZC1) holds for G.

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Partial augmentation

General difficulty: No constructions for torsion units available.

Definition

Let $u = \sum_{g \in G} z_g g$ be a torsion unit in $\mathbb{Z}G$ and denote by g^G the conjugacy class of g in G. Then $\varepsilon_g(u) = \sum_{x \in g^G} z_x$ is called **partial augmentation** of u in respect to g.

Lemma

For a torsion unit $u \in V(\mathbb{Z}G)$ there exists an $x \in U(\mathbb{Q}G)$ s.t. $x^{-1}ux \in G$ if and only if $\varepsilon_g(u^k) \geq 0$ for all $g \in G$ and $k \in \mathbb{N}$.

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General knowledge on partial augmentations

Lemma (Berman, Higman)

$u = \sum_{g \in G} z_g g$, $u \neq 1$ a torsion unit in $\mathbb{Z}G$, then $z_1 = \varepsilon_1(u) = 0$.

Lemma (Marciniak, Ritter, Sehgal, Weiss; Hertweck)

Let $u \in V(\mathbb{Z}G)$ be of finite order. If $\varepsilon_g(u) \neq 0$, then the order of g divides the order of u.

Lemma (Hertweck)

For $u \in V(\mathbb{Z}G)$ suppose that the p-part of u is conjugate in $\mathbb{Z}_p G$ to an element $x \in G$. Then $\varepsilon_g(u) = 0$, if the p-part of g is not conjugate to x. Here $\mathbb{Z}_p$ denotes the p-adic integers.

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Proof strategy

Inductive approach:
Assume $u \in V(\mathbb{Z}G)$ is a minimal counterexample to ZC, i.e every proper power of u is rationally conjugate to a group element and ZC holds for proper subgroups and quotients of G. Then a group theoretic observation yields

Theorem (del Río, Sehgal '06)

Let A be an abelian normal subgroup of G with abelian quotient. Then $\varepsilon_g(u) \geq 0$ for $g \in G \setminus C_G(A)$.

Similarly one gets:

Lemma

Let $A \trianglelefteq G$, A cyclic, G/A abelian and set $D = Z(C_G(A))$. Then $\varepsilon_g(u) \geq 0$ for $g \in G \setminus D$.

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Two cases

For a normal subgroup N of G denote by $\omega_N : \mathbb{Z}G \to \mathbb{Z}(G/N)$ the linear extension of the natural homomorphism $G \to G/N$.

Let $A \trianglelefteq G$, A cyclic, G/A abelian, $D = Z(C_G(A))$ and assume u is a minimal counterexample to ZC. Then we study separately:

- $\omega_D(u) = 1$. Using p-adic methods, especially Weiss' double action formalism and "Permutation module"-results and Cliff-Weiss Theorem (details below).
- $\omega_D(u) \neq 1$. Using the Luthar-Passi method, which relates eigenvalues under complex representations and partial augmentations (not in this talk).

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Weiss' double action formalism

Translating questions about units into questions about modules: For a group homomorphism $\alpha : H \to \mathrm{GL}_k(RG)$ the set $(RG)^k$ becomes a $R(G \times H)$-module by linearly extending the operation

$$x \cdot (g, h) = g^{-1} x \alpha(h).$$

This module is denoted M^α. If $\beta : H \to \mathrm{GL}_k(RG)$ is another group homomorphism, then $M^\alpha \cong M^\beta$ if and only if α and β are conjugate in $\mathrm{GL}_k(RG)$, i.e. there exists $u \in \mathrm{GL}_k(RG)$ s.t. $u^{-1}\alpha(h)u = \beta(h)$ for all $h \in H$.

Let $N \trianglelefteq G$ with $|G : N| = k$ and $\langle c \rangle = H \cong \langle u \rangle$ with u a torsion unit in $\mathbb{Z}G$. Then u operates on $(\mathbb{Z}N)^k = \bigcup \mathbb{Z}(g_i N) = \mathbb{Z}G$ via multiplication, so u can be seen as an element A in $\mathrm{GL}_k(\mathbb{Z}N)$. Set $\alpha : c \mapsto A$, then u is rationaly conjugate to $g \in G$ if and only if $\mathbb{Q} \otimes M^\alpha \cong \mathbb{Q} \otimes M^\beta$, where $\beta : c \mapsto g$. The character of this module is

$$\chi(g, h) = |C_G(g)| \varepsilon_g(\alpha(h)).$$

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The Matrix-Method of Marciniak, Ritter, Sehgal, Weiss

Let $\varepsilon_k : \mathrm{GL}_k(\mathbb{Z}G) \to \mathrm{GL}_k(\mathbb{Z})$ be the augmentation map applied elementwise and set $\mathrm{SGL}_k(\mathbb{Z}G) = \ker(\varepsilon_k)$. A stronger version of the Zassenhaus Conjecture is:
Is every $A \in \mathrm{SGL}_k(\mathbb{Z}G)$ of finite order conjugate in $\mathrm{GL}_k(\mathbb{Q}G)$ to a diagonal matrix with entries in G?

Theorem (Cliff, Weiss '00)

For a nilpotent group G every $A \in \mathrm{SGL}_k(\mathbb{Z}G)$ of finite order is conjugate in $\mathrm{GL}_k(\mathbb{Q}G)$ to a diagonal matrix with entries in G for every k if and only if at most one Sylow subgroup of G is non-cyclic.

If $N \trianglelefteq G$, $|G : N| = k$, then under the homomorphism $\langle u \rangle \to \mathrm{GL}_k(\mathbb{Z}N)$ described above, u is mapped into $\mathrm{SGL}_k(\mathbb{Z}N)$ if and only if $\omega_N(u) = 1$.
$\to$ Use Cliff-Weiss Theorem to prove ZC for such units, if G has a nilpotent normal subgroup with at most one non-cyclic Sylow subgroup.

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The proof

Hertweck can reduce the case $\omega_{C_G(A)}(u) = 1$ to the case $\omega_A(u) = 1$, where Cliff-Weiss is available, since A is cyclic. We use $D = Z(C_G(A))$ instead and this is in general not cyclic.

Lemma (extracted out of Cliff and Weiss' paper)

Let N be an abelian normal subgroup of G, u a torsion unit in $\mathbb{Z}G$ satisfying $\omega_N(u) = 1$, η an irreducible character of N and $n \in N$. Then

$$\sum_{h \in \ker \eta} |C_G(hn) : N| \varepsilon^G_{hn}(u) \geq 0.$$

Combining this with the following Theorem analogues to a Theorem of Hertweck, but with a more technical proof, we obtain our result. ($\mathbb{Z}_p$ denotes the p-adic integers.)

Theorem

If the order of u is the power of a prime p, then u is conjugate in $\mathbb{Z}_pG$ to an element in D.

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Thank You!

Thank you for your attention!

Enjoy the other Zassenhaus-Conjecture-talks!