Algebraic groups and complete reducibility

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Idea: Generalise notion of complete reducibility from $\mathsf{GL}_n(k)$ to arbitrary reductive algebraic groups.

I. Complete reducibility

II. A geometric approach

III. Non-algebraically closed fields

I. Complete reducibility

k a field (assume $k = \bar{k}$ for now). $\mathrm{char}(k) = p$.

Recall: A (closed) subgroup H of $\mathsf{GL}_n(k)$ is completely reducible if the inclusion $H \rightarrow \mathsf{GL}_n(k)$ is a completely reducible representation.

Let G be a reductive algebraic group over k (e.g., $\mathsf{GL}_n(k)$, $\mathsf{SO}_n(k)$, $\mathsf{Sp}_n(k)$, k^*).

Definition (Serre): Let $H \leq G$. We say H is *(G-)completely reducible* if whenever H is contained in a parabolic subgroup P of G, H is contained in some Levi subgroup L of P.

(Agrees with usual definition when $G = \mathsf{GL}_n(k)$.)

Motivation and applications

• Subgroup structure of simple algebraic groups (Liebeck, Seitz, Stewart, Testerman). Given a subgroup H of G, either H is completely reducible or it isn't! In both cases, we gain information about H.

• Maximal subgroups of finite groups of Lie type (Liebeck-M.-Shalev).

• Subcomplexes of spherical buildings.

Idea: Which properties of complete reducibility for $\mathrm{GL}_n(k)$ carry over to arbitrary G?

II. A geometric approach

R.W. Richardson: Let $N \in \mathbb{N}$. Then G acts on G^N by simultaneous conjugation: if $\mathbf{g} = (g_1, \ldots, g_N) \in G^N$ and $g \in G$ then define

$$g \cdot (g_1, \ldots, g_N) := (gg_1g^{-1}, \ldots, gg_Ng^{-1}).$$

Theorem (Richardson 1988, BMR 2005): Let $\mathbf{g} = (g_1, \ldots, g_N) \in G^N$ and let H be the closed subgroup of G generated by the g_i. Then H is completely reducible if and only if the orbit $G \cdot \mathbf{g}$ is a closed subset of G^N.

Allows us to use results from geometric invariant theory to prove results about complete reducibility.

Theorem (M 2003): Let F be a finite group. Then there are only finitely many conjugacy classes of homomorphisms $\rho\colon F \to G$ such that $\rho(F)$ is completely reducible.

Theorem (BMR 2005): If $H \leq G$ is completely reducible then $C_G(H)$ is completely reducible.

Theorem (M 2003, BMR 2005): If $H \leq G$ is completely reducible and N is a normal subgroup of H then N is completely reducible.

Theorem: If $H \leq G$ is not completely reducible then $N_G(H)$ is not completely reducible.

Proof: By the Hilbert-Mumford-Kempf-Rousseau (HMKR) Theorem, there is a **canonical** parabolic subgroup P of G such that P contains H but no Levi subgroup of P contains H. Since P is canonical, $N_G(H)$ normalizes P, so $N_G(H) \leq P$. Clearly no Levi subgroup of P contains $N_G(H)$, so $N_G(H)$ is not completely reducible.

III. Non-algebraically closed fields

Now assume G is defined over k, where we don't assume k to be algebraically closed.

Definition (Serre): Let H be a k-defined subgroup of G. We say H is *(G-)completely reducible over k* if whenever H is contained in a k-defined parabolic subgroup P of G, H is contained in some k-defined Levi subgroup L of P.

Note: H is completely reducible if and only if H is completely reducible over $\bar{k}$.

Question: Is it the case that H is completely reducible over k if and only if H is completely reducible?

McNinch 2005: No to forward direction. There exists $H \leq \mathsf{SL}_p(k)$, k nonperfect, such that H is completely reducible over k but not completely reducible. (Theory of pseudo-reductive groups.)

BMRT 2010: No to reverse direction. There exists $H \cong S_3 \leq G_2$, $p = 2$, k nonperfect such that H is completely reducible but not completely reducible over k.

Uchiyama 2012, 2013: Further counter-examples to reverse direction $p = 2$ and $G = E_6, E_7$. Systematic approach.

Geometric characterization (BHMRT 2013): Let $\mathbf{g} = (g_1, \ldots, g_N) \in G(k)^N$ and let H be the closed subgroup of G generated by the g_i. Then H is completely reducible if and only if the orbit $G(k)\cdot\mathbf{g}$ is a "cocharacter-closed" subset of G^N.

But: We do not have a rational version of the HMKR Theorem.

Open problem: If H is completely reducible over k, is $C_G(H)$ completely reducible over k?

Theorem (BMR 2010): Let H be a k-defined subgroup of G. Let k'/k be a finite Galois field extension. Then H is completely reducible over k' if and only if H is completely reducible over k.

Forward direction: Suppose H is **not** completely reducible over k'. Would like to take P to be the canonical k'-defined parabolic subgroup containing H; the canonical property should imply that P is Gal(k'/k)-stable and hence k-defined, which would imply that H is not completely reducible over k. **But:** We don't have a rational HMKR Theorem.

Instead apply the Tits Centre Conjecture for spherical buildings (proved by Tits-Mühlherr, Leeb-Ramos-Cuevas, Ramos-Cuevas).

Motivation

• Spherical buildings and complete reducibility.

• Geometric invariant theory over non-algebraically closed k (BHMR). Let V be an affine G-variety, $v \in V(k)$. How does (the closure of) $G(\bar{k}) \cdot v$ split into $G(k)$-orbits? E.g., if $v, v' \in V(k)$ are in the same $G(\bar{k})$-orbit, must they be in the same $G(k)$-orbit?

Kempf's 1978 HMKR Theorem paper has nearly 90 citations!

• Strengthened version of Tits Centre Conjecture for spherical buildings (BMR): motivated by geometric invariant theory.

• Subgroup structure of (pseudo-)reductive groups defined over k.