Sidki's Conjecture; showing finiteness of group presentations using amalgams

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Joint work with Sergey Shpectorov (University of Birmingham)

Some presentations

The following is a well-known presentation for the Alternating group A_{m+2}:

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Conjecture: This presentation is finite.

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- If $n|n'$, then $Y(m,n)$ is a quotient of $Y(m,n')$

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Define:

$$\begin{aligned} y(m,n) := \langle a, S_m : a^n = 1, [(1,2)^{a^s},(1,2)] = 1 \quad \forall s \\ (1,2)^{1+a+\cdots+a^{n-1}} = 1, \\ a^{(i,i+1)} = a^{-1} \text{ for } 2 \le i \le m-1 \rangle \end{aligned}$$

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For n odd, Sidki showed that $Y(m,n) \cong y(m,n)$.

An embedding

When $m = 3$, consider the following map:

$$a \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \quad (1,2) \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad (2,3) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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The map above can be extended to give an embedding $y(m, n) \hookrightarrow SL_{2^{m-2}}(\mathbb{F})$.

Some computational results

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The problem: Identify the group $y(m)$

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Suppose $\Omega = \{1, \ldots, m+1\}$. Let Γ be the set of all non-empty proper subsets of Ω. The type of an element is its size. Two elements are incident if one is contained in the other. Then a maximal flag contains elements of every type, so we have a geometry.

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2. *the partial multiplication restricted to G_i, for all $i \in I$, makes G_i into a group*
3. *for any $a, b \in \mathcal{A}$, the product ab is only defined when $a, b \in G_i$, for some $i \in I$*
4. *for all $i, j \in I$, $G_i \cap G_j$ is a subgroup of both G_i and G_j*

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The G_i are called the members of the amalgam.

Definition

A completion of an amalgam $\mathcal{A}$ is a pair (G, ϕ), where G is a group and $\phi : \mathcal{A} \to G$ is a map such that the restriction $\phi|_{G_i} : G_i \to G$ of ϕ to some group G_i, for $i \in I$, is a group homomorphism.

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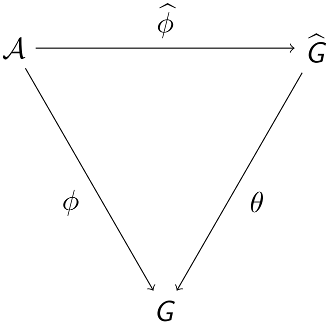
A completion $(\widehat{G}, \widehat{\phi})$ is the universal completion if given any other completion (G, ϕ), there exists a unique group homomorphism $\theta : \widehat{G} \to G$ such that



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Let $\mathcal{F}$ be a maximal flag $\{x_1\} \subset \cdots \subset \{x_1, \ldots, x_n\}$. Define G_i to be the stabiliser of the i^{th} element in the flag. So, $G_i \cong S_i \times S_{n+1-i}$. Then, $\bigcup G_i$ is an amalgam.

Tits' Lemma

How do we link the geometry and the amalgam?

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Theorem (Tits' Lemma)

Let Γ be a connected, residually connected geometry and let $G \leq Aut(\Gamma)$ be a group which acts transitively on the maximal flags of Γ. Then,

$$\Gamma \text{ is simply connected} \quad \Leftrightarrow \quad G \text{ is the universal completion of the amalgam of flag-stabilisers.}$$

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The Coxeter presentation.

$$S_{m+1} = \langle a_1, \ldots, a_m | a_i^2 = 1, (a_i a_{i+1})^3 = 1 \ \forall i, (a_i a_j)^2 = 1, \forall i \neq j \rangle$$

Back to Sidki's problem

We want to have an amalgam for the group

$$y(m) := \langle a, S_m : [(1,2)^{a^s}, (1,2)] = 1 \quad \forall s \\ a^{(i,i+1)} = a^{-1} \text{ for } 2 \leq i \leq m-1 \rangle$$

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Define

- $G_1 = \langle s_1, \ldots, s_{m-1} \rangle \cong S_m$
- $G_i = \langle a, s_1, \ldots, \hat{s_{i-1}}, \ldots, s_{m-1} \rangle \cong y(i-1) : S_{m-i}$

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Let $\tilde{y}(m) = \langle \tau, a, \tau s_1, \dots \tau s_{m-1} \rangle \cong \langle \tau \rangle y(m)$

- $G_1 = \langle \tau, \tau s_1, \dots, \tau s_{m-1} \rangle \cong 2 \times S_m$
- $G_i = \langle \tau, a, \tau s_1, \dots, \widehat{\tau s_{i-1}}, \dots, \tau s_{m-1} \rangle \cong \tilde{y}(i-1) : S_{m-i}$

But before we do, one more thing:

There exists an involution τ which acts on $y(m)$ by centralising the s_i and inverting a.

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$\mathcal{A} = \bigcup_{i=1}^{m} G_i$

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Recall that $\mathbb{F}$ is a field of characteristic 2. Hence, $(\cdot,\cdot)$ is a symplectic form.

$$\text{i.e. } (u,u) = 0 \text{ for all } u \in V$$

Choosing forms

Assume that V has a basis $u, v_1, \ldots, v_m$ and $S_m < \tilde{y}(m)$ acts naturally by permuting the v_i.

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$$\left(\begin{array}{c|ccccc} 0 & \alpha & \alpha & \ldots & \alpha & \alpha \\ \hline \alpha & 0 & 1 & \ldots & 1 & 1 \\ \alpha & 1 & 0 & \ldots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha & 1 & 1 & \ldots & 0 & 1 \\ \alpha & 1 & 1 & \ldots & 1 & 0 \end{array}\right)$$

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We may scale u to get $\alpha = 1$.

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So we choose $q(v_i) = t^{-1}$ and $\mathbb{F} = \mathbb{F}_2[t, t^{-1}]$.

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Thank you for listening!

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