Centralizer-like subgroups associated with words in two variables

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6th August 2013

Groups St Andrews 2013

Verbal and marginal subgroups

Let $w(x_1, \dots, x_n)$ be a word and let G be a group. Then

- $w(G) = \langle w(g_1, \dots, g_n) \mid g_1, \dots, g_n \in G \rangle$ is termed the verbal subgroup of G determined by w;
- the ith partial margin, $1 \le i \le n$, of w in G is the subgroup $w_i^*(G)$ of all elements $a \in G$ with

$$w(g_1, \dots, g_i a, \dots, g_n) = w(g_1, \dots, g_i, \dots, g_n)$$

  for all $g_1, \dots, g_n \in G$;
- the margin of w in G is given by $w^*(G) = \bigcap_{1 \le i \le n} w_i^*(G)$.

Example

If $w(x_1, x_2) = [x_1, x_2]$, then

$$w(G) = G' \text{ and } w^*(G) = w_1^*(G) = w_2^*(G) = Z(G).$$

Let $w(x, y)$ be a two-variable word and let g be an element in the group G. Then, we consider

$$w_2^*(g) = \{a \in G \mid w(g, ha) = w(g, h) \ \forall h \in G\},$$
$${}^*w_2(g) = \{a \in G \mid w(g, ah) = w(g, h) \ \forall h \in G\}.$$

Proposition (L.C. Kappe, P.M. Ratchford)

- $w_2^*(g)$ and ${}^*w_2(g)$ are subgroups of G;
- $w_2^*(G) = \bigcap_{g \in G} w_2^*(g)$ and $w_2^*(G) = \bigcap_{g \in G} {}^*w_2(g)$;
- if $w(x, y) = [x, y]$, then

$$w_2^*(g) = C_G(g^G) \text{ and } {}^*w_2(g) = C_G(g).$$

Let $w(x, y)$ be a two-variable word and let g be an element in the group G. Then, we define

$$w_2^*(g) = \{a \in G \mid w(g, ha) = w(g, h)\ \forall h \in G\},$$
$$^*w_2(g) = \{a \in G \mid w(g, ah) = w(g, h)\ \forall h \in G\},$$

$$w_1^*(g) = \{a \in G \mid w(ha, g) = w(h, g)\ \forall h \in G\},$$
$$^*w_1(g) = \{a \in G \mid w(ah, g) = w(h, g)\ \forall h \in G\}.$$

They have been called centralizer-like subgroups associated with the word w.

Let $w(x, y)$ be a two-variable word. For every $g \in G$ we consider the following centralizer-like subsets associated with the word w:

$$W_L^w(g) = \{a \in G \mid w(g, a) = 1\},$$
$$W_R^w(g) = \{a \in G \mid w(a, g) = 1\}.$$

Observation (L.C. Kappe, P.M. Ratchford)

- If $w(g, 1) = 1$, then $w_2^*(g) \subseteq W_L^w(g)$ and ${}^*w_2(g) \subseteq W_L^w(g)$;
- if $w(1, g) = 1$, then $w_1^*(g) \subseteq W_R^w(g)$ and ${}^*w_1(g) \subseteq W_R^w(g)$.

Question

Are $W_L^w(g)$ and $W_R^w(g)$ always subgroups of G?

Let $\mathcal{Y}$ be the class of groups which cannot be covered by conjugates of any proper subgroup, i.e. if H is a subgroup of a $\mathcal{Y}$-group G, then

$$G = \bigcup_{g \in G} H^g \implies G = H.$$

Theorem (M. Herzog, P. Longobardi, M. Maj)

A $\mathcal{Y}$-group G is abelian if for every pair of elements $a, b \in G$ there exists $g \in G$ for which $[a^g, b] = 1$.

Counterexamples

- S.V. Ivanov constructed, for every sufficiently large prime p, a 2-generator group covered by the conjugates of a subgroup of order p;
- G. Cutolo, H. Smith, J. Wiegold showed that there are (non-abelian) groups of finitary permutations which are the union of the conjugates of an abelian subgroup.

On a property of two-variable laws

M. Meriano, C. Nicotera, "On certain weak Engel-type conditions in groups", to appear in Communications in Algebra.

Question

Let G be a $\mathcal{Y}$-group and $w(x, y)$ be a word;
if for every pair of elements $a, b \in G$ there exists $g \in G$ such that

$$w(a^g, b) = 1,$$

then is it true that $w(a, b) = 1$ for all $a, b \in G$,
i.e. does G belong to the variety determined by the law $w(x, y) = 1$?

Proposition

Let G be a $\mathcal{Y}$-group, $w(x, y)$ a word in two variables, and assume that for all $a, b \in G$ there exists $g \in G$ such that $w(a^g, b) = 1$. G belongs to the variety determined by $w(x, y)$ if one of the following conditions holds:

i) $W_L^w(g)$ is a subgroup of G for every $g \in G$,

ii) $W_R^w(g)$ is a subgroup of G for every $g \in G$.

The classes $\mathscr{W}_L^w$ and $\mathscr{W}_R^w$

Let $w(x,y)$ be a word and denote by $\mathscr{W}_L^w$ the class of groups G for which the set $W_L^w(g)$ is a subgroup of G for every $g \in G$.

Proposition

Let w be a two-variable word and assume that the following two conditions are satisfied:

i) $w(g,1) = 1$, for every $g \in G$;

ii) for every $g, h, k \in G$ there exist $c_1, c_2 \in G$ for which

$$w(g, hk) = w(g,h)^{c_1} w(g,k)^{c_2}.$$

Then G belongs to the class $\mathscr{W}_L^w$.

We denote by $\mathscr{W}_R^w$ the class of groups G for which the set $W_R^w(g)$ is a subgroup of G for every $g \in G$.

Some commutator words in two variables

In 1966, N.D. Gupta considered a number of group laws of the form

$$c_n = [x, y],$$

where c_n is a commutator of weight $n \geq 3$ with entries from the set consisting of x, y and their inverses.

Example

Let $[y, x, x] = [x, y]$. Then

$$\begin{aligned} [y, x, x] = [x, y] \quad &\Leftrightarrow \quad [y, x]^{-1}[y, x]^x = [y, x]^{-1} \\ &\Leftrightarrow \quad [y, x]^x = 1 \;\; \Leftrightarrow \;\; [y, x] = 1. \end{aligned}$$

Therefore if $[y, x, x] = [x, y]$ is a law in a group G, then G is abelian.

Problem

Is each group satisfying one of the laws $c_n = [x, y]$ abelian?

Theorem (N.D. Gupta)
Any finite or solvable group satisfying one of the laws $c_n = [x, y]$ is abelian.

Theorem (L.C. Kappe, M.J. Tomkinson)
The variety of groups satisfying one of the laws $c_3 = [x, y]$ is the variety of all abelian groups.

Theorem (P. Moravec)
The variety of groups satisfying one of the laws $c_4 = [x, y]$ is the variety of all abelian groups.

Let

$$v(x, y) = C_n[y, x],$$

where C_n is a left-normed commutator of weight $n \geq 3$ with entries drawn from the set $\{x, y, x^{-1}, y^{-1}\}$. We denote by $\mathcal{F}_n$ the set of words $C_n[y, x]$.

Under what conditions does a group belong to the classes $\mathscr{W}_L^v$ and $\mathscr{W}_R^v$ associated with the word $v(x, y) = C_n[y, x]$?

Definition

We say that two n-variable words w_1 and w_2 are strongly equivalent in a group G if, for every $g_1, \ldots, g_n$ in G,

$$w_1(g_1, \ldots, g_n) = 1 \Leftrightarrow w_2(g_1, \ldots, g_n) = 1.$$

Proposition

Let G be a nilpotent group and $v(x, y) \in \mathcal{F}_n$, $n \geq 3$. Then $v(x, y)$ is strongly equivalent to $[x, y]$ in G.

If $n = 3$, then there are 32 non-trivial words of the form

$$[r, s, t][y, x],$$

where $r, s, t \in \{x, y, x^{-1}, y^{-1}\}$.

We refer to the two-variable word $[r, s, t][y, x]$ as $v_i(x, y)$, where the integer i is the number of the Table corresponding to $[r, s, t]$.

Table:

1. $[x, y, y]$	9. $[x^{-1}, y, y]$	17. $[y, x, x]$	25. $[y^{-1}, x, x]$
2. $[x, y, y^{-1}]$	10. $[x^{-1}, y, y^{-1}]$	18. $[y, x, x^{-1}]$	26. $[y^{-1}, x, x^{-1}]$
3. $[x, y^{-1}, y]$	11. $[x^{-1}, y^{-1}, y]$	19. $[y, x^{-1}, x]$	27. $[y^{-1}, x^{-1}, x]$
4. $[x, y^{-1}, y^{-1}]$	12. $[x^{-1}, y^{-1}, y^{-1}]$	20. $[y, x^{-1}, x^{-1}]$	28. $[y^{-1}, x^{-1}, x^{-1}]$
5. $[x, y, x]$	13. $[x^{-1}, y, x]$	21. $[y, x, y]$	29. $[y^{-1}, x, y]$
6. $[x, y, x^{-1}]$	14. $[x^{-1}, y, x^{-1}]$	22. $[y, x, y^{-1}]$	30. $[y^{-1}, x, y^{-1}]$
7. $[x, y^{-1}, x]$	15. $[x^{-1}, y^{-1}, x]$	23. $[y, x^{-1}, y]$	31. $[y^{-1}, x^{-1}, y]$
8. $[x, y^{-1}, x^{-1}]$	16. $[x^{-1}, y^{-1}, x^{-1}]$	24. $[y, x^{-1}, y^{-1}]$	32. $[y^{-1}, x^{-1}, y^{-1}]$

Proposition

Let G be a group, $g \in G$, and $v_i, v_j \in \mathfrak{F}_3$. Then the following hold:

- $v_i(x, y)$ is strongly equivalent to $[x, y]$, if $i \in \{17, 18, 19, 21, 22, 29\}$;
- $v_i(x, y)$ is strongly equivalent to $v_j(x, y)$, if $(i, j) = (2, 3)$ or $(i, j) = (6, 13)$;
- $v_i(x, y)$ is strongly equivalent to $v_i(y, x)$, if $i \in \{7, 9\}$;
- $v_i(x, y)$ is strongly equivalent to $v_j(y, x)$, if $(i, j) = (1, 5)$ or $i \in \{2, 3\}$ and $j \in \{6, 13\}$.

Proposition

Consider $v_i(x, y) = [r, s, t][y, x] \in \mathfrak{F}_3$, and let $\overline{v}_i(y, x) = [s, r, t][x, y]$. If G is a metabelian group, then for every $g \in G$ we have the following:

- $v_i(x, y)$ is strongly equivalent to $\overline{v}_i(y, x)$;
- $W_L^{v_i}(g) = W_R^{\overline{v}_i}(g)$ and $W_R^{v_i}(g) = W_L^{\overline{v}_i}(g)$.

Proposition

Let $w(x,y) = v_i(x,y)$ in $\mathfrak{F}_3$. Then the following hold:

- if $i \in \{17, 18, 19, 21, 22, 29\}$, then any group G belongs to the classes $\mathscr{W}_L^w$ and $\mathscr{W}_R^w$;
- if $i \in \{5, 6, 13, 14, 20\}$, a metabelian group belongs to $\mathscr{W}_L^w$ but not necessarily to $\mathscr{W}_R^w$, and there are solvable groups G with $G'' \neq 1$ which do not belong to $\mathscr{W}_L^w$;
- if $i \in \{1, 2, 3, 4, 30\}$, a metabelian group belongs to $\mathscr{W}_R^w$ but not necessarily to $\mathscr{W}_L^w$, and there are solvable groups G with $G'' \neq 1$ which do not belong to $\mathscr{W}_R^w$;
- if $w(x,y)$ is one of the remaining words, then a group, even a metabelian one, does not necessarily belong to $\mathscr{W}_L^w$ or $\mathscr{W}_R^w$.

Thank you!