Simplicity result for groups acting on trees

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Property (P)

Let G be a group acting on a tree T.

Definition

For a path L in T define $\mathrm{pr}_L : VT \to VT$ such that $\mathrm{pr}_L(v)$ is the vertex on L that is closest to v.
For a vertex x in L denote with $G^x_{(L)}$ the permutation group induced by $G_{(L)}$ on $\mathrm{pr}_L^{-1}(x)$.
The action is said to have property P *if the homomorphism*

$$G_{(L)} \to \prod_{x \in L} G^x_{(L)}$$

is an isomorphism for all paths L (finite or infinite) in T.

Tits's Theorem

Theorem

(Tits, 1970) Let T be a tree and G a subgroup of $\mathrm{Aut}(T)$. Assume that G does not stabilize a proper non-empty subtree, and that G does not fix an end of T. Assume furthermore that the action of G on T has property P. Then G^+, the subgroup of G generated by stabilizers of edges, is simple.

Property H

Let $e = \{v, w\}$ be an edge in a tree T. If both components of $T \setminus e$ are infinite then each of them is called a *half-tree*.

Let $G < \mathrm{Aut}(T)$.
We say the action has *property H* if the stabilizer of every half-tree in T is non-trivial.

Simplicity result.

Theorem

(M.&V., 2012) Let T be a tree and G a closed subgroup of $\mathrm{Aut}(T)$*. Assume that G does not stabilize a proper non-empty subtree, and that G does not fix an end of T. Assume furthermore that the action of G on T has property H. Then G^{++}, the subgroup of G generated by pointwise stabilizers of half-trees, is topologically simple.*

If N is a closed non-trivial subgroup of G normalized by G^{++} then $G^{++} \leq N$.

Primitive graphs

Definition

A group G is said to act primitively *on a set Y if no non-trivial proper equivalence relation on Y is preserved by the action.*

Definition

A graph X is said to be primitive *if the automorphism group acts primitively on the vertex set VX.*

Simplicity for automorphism groups of primitive graphs

Theorem

(M.&V., 2012) Let G be the automorphism group of a locally finite primitive graph X with more than one end. Then G has a topologically simple subgroup of finite index.

Ingredients of proof

Theorem

(M., 1994) Let X be a locally finite primitive graph with more than one end. Set $G = \mathrm{Aut}(X)$. Then there is a pair of vertices v, w such that the graph with the same vertex set as X and edge set $G\{v, w\}$ has connectivity 1.

Theorem

(Simon M. Smith, 2006) If G acts primitively on a locally finite graph with more than one end then the vertex stabilizers are infinite.

Great minds think alike

Down Under, in Newcastle, Australia, Chris Banks, Murray Elder and George Willis have been barking up the next tree.

Property P^k

Definition

(Banks, Elder and Willis, 2013) For an edge $\{v, w\}$ in T let $B_k(v, w)$ denote the set of all vertices that are in distance at most k from either v or w.
A group $G < \mathrm{Aut}(T)$ is said to have property P^k *if for every edge $\{v, w\}$ in T the group $G_{(B_k(v,w))}$ acts independently on the two half trees defined by the edge $\{v, w\}$.*

Simplicity theorem

Theorem

(Banks, Elder and Willis, 2013) Let G be a closed subgroup of $\mathrm{Aut}(T)$ *that does not stabilize a proper non-empty subtree of* T *and fixes no end of* T. *Assume that* G *has property* P^k. *Let* G^{+_k} *denote the subgroup generated by the groups* $G_{(B_k(v,w))}$ *for all edges* $\{v, w\}$ *in* T. *Then* G^{+_k} *is simple or trivial.*

Improvements

- Neither simplicity theorem contains the other.
- Using the result of Banks et al. one can get a stronger result about the automorphism groups of primitive graphs with infinitely many ends:

Theorem

Let G be the automorphism group of a locally finite primitive graph X with more than one end. Then G has a simple subgroup of finite index.