On the Covering Number of Some Symmetric Groups

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Abstract

- According to Bernhard Neumann, every group with a non-cyclic finite homomorphic image is the union of finitely many proper subgroups. The minimal number of subgroups needed to cover a group G is called the *covering number* of *G.*

- Tomkinson showed that for a solvable group the covering number is $p^{\alpha} +1$, where p is a prime number , and he suggested the investigation of the covering numbers for families of finite simple groups. So far, a few results are known, among them some for small alternating groups, several types of linear groups, and the *Suzuki* groups.

- For sporadic simple groups the covering numbers are known for the *Mathieu* groups M_{11}, M_{22} and M_{23}, as well as *Ly* and *O'N*. Furthermore, estimates have been given for *J1* and M_cL. We are also working on the covering number of the *Mathieu* group M_{12}.

Introduction

- It is well known that no group is the union of *2* proper subgroups.
- The question which integers can be covering numbers of groups was raised by Tomkinson.
- It was proved so far that there are no groups with covering number 2, 7, 11, 19, 21, 22, 25 (Tomkinson, Detomi, Lucchini,…), and the smallest integer for which it is not known whether it is a covering number for a group is

  ***n* = 27**.

The Covering Number

- **Theorem 1 (Tomkinson,1997):** *Let G be a finite soluble group and let p^{α} be the order of the smallest chief factor having more that one complement. Then* $\boldsymbol{\sigma(G) = p^{\alpha} + 1}$.

- The author suggested the investigation of the covering number of simple groups.

Linear Groups

- **Theorem 2 (Bryce, Fedri, Serena, 1999)**
- *$\sigma(G)=1/2\ q(q+1)$ when q is even,*
- *$\sigma(G)=1/2\ q(q+1)+1$ when q is odd,*

where G=PSL(2,q), PGL(2,q), or GL(2,q),

and $q \neq 2, 5, 7, 9$.

Suzuki Groups

- **Theorem 3 (Lucido, 2001)**
- $\sigma(Sz(q)) = \frac{1}{2} q^2(q^2+1)$,

  where $q = 2^{2m+1}$.

Sporadic Simple Groups

Theorem 6 (P.E. Holmes, 2006)

$\sigma(m_{11})=23$, $\sigma(m_{22})= 771$, $\sigma(m_{23})=41079$,

$\sigma(Ly) = 112845655268156$,

$\sigma(O'N) = 36450855$

$5165 \leq \sigma(J_1) \leq 5415$

$24541 \leq \sigma(M^CL) \leq 24553$.

The author has used *GAP*, the *ATLAS*, and Graph Theory.

Symmetric and Alternating Groups

- **Theorem 4** (Maroti, 2005)
- $\sigma(S_n) = 2^{n-1}$ *if n is odd*, $n \neq 9$
- $\sigma(S_n) \leq 2^{n-2}$ *if n is even.*
- $\sigma(A_n) \geq 2^{n-2}$ *if* $n \neq 7,9$, *and* $\sigma(A_n) = 2^{n-2}$ *if n is even but not divisible by 4.*
- $\sigma(A_7) \leq 31$, *and* $\sigma(A_9) \geq 80$.

Alternating Groups

- **Theorem 5 (Luise-Charlotte Kappe, Joanne Redden, 2009)**
- $\sigma(A_7) = 31$
- $\sigma(A_8) = 71$
- $127 \le \sigma(A_9) \le 157$
- $\sigma(A_{10}) = 256.$

Small Symmetric Groups

- We started to determine the exact covering number of some of the first symmetric groups: S_8, S_9, S_{10}.

Recent results

- We can now prove the exact numbers:
- **$\sigma(S_8) = 64$.**
- **$\sigma(S_{10})=221$.**
- We still have a range for S_9:
- **$242<\sigma(S_9)<257$.**

Our Strategy

- It is sufficient to consider the number of maximal subgroups of G needed to cover all maximal cyclic subgroups of G.
- We use *GAP* for the distribution of the elements in the maximal subgroups
- We estimate the limits by a *Greedy Algorithm.*

Strategy

- **Easy case**: When the elements are partitioned into the subgroups of a conjugacy class
- **Harder case**: When the elements of a certain cyclic structure are not partitioned.
- **Aproach:** Greedy algorithm + Graph Theory Algorithms for minimum coverings.
- **New Approach:** Incidence matrices and Combinatorics

S7

- It is clear from the table why $\sigma = 2^{7-1}$
- The group is covered by A_7 (MS1), the 7 groups S_6 in MS2, the 35 groups in MS3, and the 21 groups in MS4: $1+7+35+21=64=2^6$.

- $\sigma(S7) = 64$.

Maximal subgroups	**Order of Class Representative**	**Size**
MS1 = A_7	2520	**1**
MS2 = S_6	720	**7**
MS3 = S_3 x S_4	144	**35**
MS4 = C_2 x S_5	240	**21**
MS5 = (C_7:C_3):C_2	**42**	**120**

Distribution of Elements:

Order	**Cyclic Structure**	**Size**	**MS1=A_7**	**MS2**	**MS3**	**MS4**	**MS5**
1	1	1	1				
2	(12)	21	0	15	9	11	**0**
2	(12)(34)		X				
2	(12)(34)(56)	105	0	15,P	9	15	**7**
3	(123)		X				
3	(123)(456)		X				
4	(1234)	210	0	90	6,P	30	**0**
4	(1234)(56)		X				
5	(12345)		X				
6	(123456)	840	0	120,P	0	0	**14**
6	(123)(45)	420	0	120	36	40	**0**
6	(123)(45)(67)		X				
7	(1234567)	720	X				
10	(12345)(67)	504	0	0	0	24,P	**0**
12	**(1234)(567)**	**420**	**0**	**0**	**12,P**	**0**	**0**

S8

- Here are the maximal subgroups and the distribution of the elements of S_8 in the representatives of the maximal subgroups. In parentheses the small numbers mean in how many representatives each element is to be found.
- Example: Each element of order 6 of type 2x3 i.e. (1,2)(3,4,5) is to be found in 3 representatives of MS4, and in each representative of MS4 there are 420 such elements.
- The group is covered by A_8 (MS1), the 28 groups in MS3, and the 35 groups in MS6, i.e. 1+28+35 = 64 = 2^6.
- σ(S8) = 64.

S8

Maximal subgroups	**Order of Class Representative**	**Size**
MS1 = A_8	20160	**1**
MS2 = S_3 x S_5	720	**56**
MS3 = C_2 x S_6	1440	**28**
MS4 = S_7	5040	**8**
MS5=((((C_2xD_8):C_2):C_3):C_2):C_2	384	**105**
MS6 = (S_4 x S_4): C_2	1152	**35**
MS7 = PSL(3,2): C_2	**336**	**120**

S8

Distribution of Elements:

Order	Cyclic Structure	Size	MS1	MS2	MS3	MS4	MS5	MS6	MS7
1	1	1	1	1	1	1	1	1	**1**
2	2^1	28	0	13(26)	16(16)	21(6)	4(15)	12(15)	**0**
2	2^2	210	210, P	45(12)	60(8)	105(4)	18(9)	42(7)	**0**
2	2^3	420	0	45(6)	60(4)	105(2)	28(7)	36(3)	**28(8)**
2	2^4	105	105, P	0	15(4)	0	25(25)	33(11)	**21(24)**
3	3^1	112	112, P	22(11)	40(10)	70(5)	0	16(5)	**0**
4	2x4	2520	2520,P	90(2)	180(2)	630(2)	24,P	72,P	**0**
4	4^1	420	0	30(4)	90(6)	210(4)	12(3)	12,P	**0**
4	2^2x 4	1260	0	0	90(2)	0	36(3)	180(5)	**0**
4	4^2	1260	1260,P	0	0	0	60(5)	108(3)	**42(4)**
5	5	1344	1344,P	24,P	144(3)	504(3)	0	0	**0**
6	2x3	1120	0	100(5)	160(4)	420(3)	0	96(3)	**0**
6	2x2x3	1680	1680,P	90(3)	120(2)	210,P	0	48,P	**0**
6	$2x3^2$	1120	0	40(2)	40,P	0	32(3)	0	**0**
6	6	3360	0	0	120,P	840(2)	32,P	0	**56(2)**
6	2 x 6	3360	3360,P	0	120,P	0	32,P	192(2)	**0**
7	7	5760	5760,P	0	0	720,P	0	0	**48,P**
8	8	5040	0	0	0	0	48,P	144,P	**84(2)**
10	2 x 5	4032	0	72,P	144,P	504,P	0	0	**0**
12	3 x 4	3360	0	60,P	0	420,P	0	96,P	**0**
15	**3 x 5**	**2688**	**2688,P**	**48,P**	**0**	**0**	**0**	**0**	**0**

S9

- Here is the distribution of the elements of S_9 in the representatives of the maximal subgroups. Here is how the lower and the upper bound are clearly to be seen:
- We definitely need:
- MS1=A_9 (1 group)
- MS2 (126 groups) to cover the elements of order 20.
- MS4 (36 groups) to cover the elements of order 14, and 12.
- MS5 (9 groups) to cover the elements of order 8, and ((1,2,3)(4,5,6)(7,8).
- Then, if you take all the 84 groups of MS3, we'll cover 3 types of elements of order 6. So, 84 more groups add up to **256: the upper bound.**
- **The lower bound.:**
- If we cover the elements of type 3x6 (20160) by groups from MS6 instead (where they are not partitioned), we would have needed at least 20160/288=70 groups. So, **1+126+36+9+71=243 ≥ σ**
- **Hence, 243 ≤ σ ≤ 256.**

S9

Maximal subgroups (1376)	**Order of Class Representative**	**Size**
MS1 = A_9	181440	**1**
MS2 = S_4 x S_5	2880	**126**
MS3 = S_3 x S_6	4320	**84**
MS4 = C_2 x S_7	10080	**36**
MS5= S_8	40320	**9**
MS6 = ((((C_3x((C_3xC_3):C_2)):C_2):C_3):C_2):C_2	1296	**280**
MS7 = (((C_3xC_3):Q_8):C_3):C_2	**432**	**840**

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Distribution of Elements:

Order	Cyclic Structure	Size	MS1	MS2	MS3	MS4	MS5	MS6	MS7
1	1	1	1	1	1	1	1	1	**1**
2	2^1		0						
2	2^2								
2	2^3		0						
2	2^4								
3	3^1								
3	3^2								
3	3^3								
4	2x4	7560	7560,P						
4	4^1	756	0	36(6)	90(10)	210(10)	420(5)	0	**0**
4	2^2x 4	11340	0	180(2)	270(2)	630(2)	1260,P	162(4)	**0**
4	4^2 =8^2								
5	5	3024	3024,P						
6	2x3	2520	0	220(11)	270(9)	490(7)	1120(4)	36(4)	**0**
6	2^2x3	7560	7560,P						
6	2x3^2	10080	0	160(2)	360(3)	280,P	1120,P	36, P	**0**
6	6	10080	0	0	120,P	840(3)	3360(3)	36, P	**56(2)**
6	2 x 6	30240	30240,P						
6	$2^3$3	2520	0	60(3)	30, P	210(3)	0	36(4)	**0**
6	3x6	20160	0	0	240,P	0	0	288(4)	**72(3)**
7	7	25920	25920,P						
8	8	45360	0	0	0	0	5040,P	0	**108(2)**
9	9	40320	40320,P						
10	2 x 5	18144	0	144,P	432(2)	1008(2)	4032(2)	0	**0**
10	$2^2$5	9072	9072,P						
12	3 x 4	15120	0	360	180,P	420,P	3360	0	**0**
14	2x7	25920	0	0	0	720,P	0	0	**0**
15	3 x 5	24192	24192,P						
20	**4x5**	**18144**	**0**	**144,P**	**0**	**0**	**0**	**0**	**0**

The Covering Number of S_{10}

- To determine a minimal covering by maximal subgroups, it suffices to find a minimal covering of the conjugacy classes of maximal cyclic subgroups by maximal subgroups of the group.

Maximal subgroups

Maximal subgroups (3977)	**Order of Class Representative**	**Size**
MS1 = A_10	1814400	**1**
MS2=S_4 x S_6	17280	**210**
MS3 = S_3 x S_7	30240	**120**
MS4 = C_2 x S_8	80640	**45**
MS5 = S_9	362880	**10**
MS6= C_2 x (((C_2xC_2xC_2xC_2):A_5):C_2	3840	**945**
MS7 = (S_5 x S_5):C_2	28800	**126**
MS8 = (A_6.C_2):C_2	**1440**	**2520**

Distribution of elements generating maximal cyclic subgroups:

Order	Cyclic Structure	Size	MS1	MS2	MS3	MS4	MS5	MS6	MS7	MS8
ODD										
4	2^2x 4	56700	0	1080_4	1890_4	3780_3	11340_2	180_3	900_2	**0**
4	2x4^2	56700	0	540_2	0	1260,P	0	300_5	1800_4	**90_4**
6	2^3x3	25200	0	480_4	840_4	1680_3	2520, P	0	600_3	**0**
6	2x3^2	50400	0	1200_5	1680_4	2240_2	10080_2	160_3	800_2	**0**
6	2^2x6	75600	0	360,P	0	3360_2	0	240_3	2400_4	**0**
6	3x6	201600	0	960,P	1680,P	0	20160,P	0	0	**240_3**
8	8	226800	0	0	0	5040,P	45360	240,P	0	**180**
10	10	362880	0	0	0	0	0	384,P	2880,P	**144,P**
12	$3^2$4	50400	0	240,P	840_2	0	0	160_3	0	0
14	2x7	259200	0	0	2160,P	5760,P	25920,P	0	0	**0**
20	4x5	181440	0	964,P	0	0	18144,P	0	1440,P	**0**
30	2x3x5	120960	0	0	1008,P	2688,P	0	0	960,P	**0**
-EVEN------	----------	---------	--------	---------	--------	---------	---------	-------	--------	**------**
6	2 x 6	151200	P	720, P	2520_2	6720_2	30240_2	160 P	0	**0**
9	9	403200	P	0	0	0	P	0	0	**0**
12	4x6	151200	P	720, P	0	0	0	160, P	2400_{x2}	**0**
12	2x3x4	151200	P	1440_2	2520_2	3360 P	15120 P	0	1200 P	**0**
21	3 x 7	172800	P	0	1440 P	0	0	0	0	**0**
8	**8x2**	**226800**	**P**	**0**	**0**	**5040,P**	**0**	**240,P**	**3600_2**	**180_2**

S10

- We first found that the Covering number has **upper bound**: MS1+MS3+MS5+MS7 =1+120+10+126=257.
- However, we ran a Greedy algorithm on MS3 and found out that 84 groups only from MS3 are sufficient to cover the elements of type $3^2 4$. So:
- σ ≤ 1+84+10+126=221.
- The upper bound was reduced.
-
- **The lower bound:** The elements of type 3^2 x 4 are 50400. If they were partitioned in MS3 we would have needed 50400/840 = 60.
- So, we need at least 61 from them.
- 1+61+10+126= 198.
-
- **Hence 198 ≤ σ ≤221.**
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The covering number is 221.
Sketch of the Proof:

- It is not difficult to see from the Inventory that the groups from MS3, MS5, and MS7 represent a covering of the odd permutations, and MS1={A_{10}} covers the even.
- We want to minimize this covering.
- The problematic elements are of structure 3x3x4, of order 12.
- The proof further involves *Incidence matrices*, and *Combinatorics.*

Incidence matrices

- Let V, and U are two collections of objects. Call the objects in V elements, and the objects in U sets.
- The incidence structure between U and V can be represented by the incidence matrix A(a_{ij}) of (V,U):
- $a_{ij}=\begin{cases} 1 \; if \; v_i \in U_j \\ 0 \; if \; v_i \neg\in U_j \end{cases}$
- Let W be a sub-collection of U. We define a vector $x(W) = (x_1, x_2, \ldots x_{|U|})^T$ as follows
- $x_j=\begin{cases} 1 \; if \; u_j \in W \\ 0 \; if \; u \neg\in W \end{cases}$
- Let $A * x(W) = y(W) = (y_1, y, \ldots y_{|V|})^T, where \; y_i \geq 0.$

- If y_i=0, then $v_i \neg\in \bigcup_{u \epsilon W}$, and
- if $y_i > 0, \forall i,$ then every v_i is contained in at least one member of W. **We say that W covers V.**

- **Our goal is to minimize |W|, s.t. W covers V, i.e. maximize the number of the 0-entries in $x(W)$.**

The elements of type 3*3*4

- There are 50,400 elements of type 3*3*4 in S_{10}. They are to be found in MS3, but are not partitioned.
- Each class of MS3 contains 840 such elements, and each element is in exactly 2 subgroups of MS3.
- Because the subgroups of MS3 are isomorphic to $S_3 x S_7$, we can label them by the letters fixed by the respective S_7, i.e.
- MS3 = {H(k_1,k_2,k_3), k_1,k_2,k_3 ϵ {0,1,2,3,4,5,6,7,8,9}, k_1<k_2<k_3}.
- So, our incidence matrix will contain 120 columns, labeled by the members of MS3.
- The rows are the maximal cyclic subgroups generated by our elements. There are 6 cyclic subgroups of order 12 in the intersection of H(i_1, i_2, i_3) and H(i_4, i_5, i_6) generated by:
- $(i_1, i_2, i_3)(i_4, i_5, i_6)(i_7, i_8, i_9, i_{10})$,
- $(i_1, i_3, i_2)(i_4, i_5, i_6)(i_7, i_8, i_9, i_{10})$,
- $(i_1, i_2, i_3)(i_4, i_5, i_6)(i_7, i_9, i_8, i_{10})$,
- $(i_1, i_3, i_2)(i_4, i_5, i_6)(i_7, i_9, i_8, i_{10})$,
- $(i_1, i_2, i_3)(i_4, i_5, i_6)(i_7, i_8, i_{10}, i_9)$,
- $(i_1, i_3, i_2)(i_4, i_5, i_6)(i_7, i_8, i_{10}, i_9)$,

- and each one of them contains 4 elements of type 3*3*4: thus the 50,400 elements of type 3*3*4 are partitioned into 50,400/24=2100 equivalence classes. Our incidence matrix will have 2100 rows.

Proposition, confirming the result of the Greedy algorithm

- **Proposition:**
- Let $U = \{u=(k_1, k_2, k_3), k_1, k_2, k_3 \in \{0,1,2,3,4,5,6,7,8,9\}, k_1<k_2<k_3\}$, and
- $V=\{(u, u'); u, u' \in U, u \cap u'=\varnothing\}$. We say that $v \in u_j$ iff $u=u_j$, or $u'=u_j$.
- For this incidence relation, there exists a sub-collection W of U with $|W|=84$, which covers V, and this is a minimal covering.
- Specifically, W can be chosen as U-D, where
- $D = \{(0,k_2,k_3), k_2, k_3 \in \{1,2,3,4,5,6,7,8,9\}, k_2<k_3\}$.

- *Proof:*
- We have an incidence (0 -1) matrix A of size 2100 x 120 with exactly 2 entries equal to 1 in each row. With $x(U) = (1,1, \dots 1)^T$ we have
- y(W)=$A * x(U) = (2,2, \dots , 2)^T$.
- We want to determine the maximum numbers of 0-s entries contained in a *x(W)* vector, so that the *y(W)* vector has all non-zero entries. **We can achieve that by removing the maximal subset$\{\boldsymbol{u_1}, \boldsymbol{u_1}, \dots \boldsymbol{u_1}\}$ of U with pairwise non-trivial intersection.**

Combinatorics

- **THEOREM (Erdos, Ko, Rado): The maximal number m of k-subsets $A_{1,}$ $A_2, \ldots A_m$ of an n-set S that are pairwise non-disjoint is $m \leq \binom{n-1}{k-1}$.**
- **The upper bound is best possible, and it is attained when A_i are precisely those k-subsets of S which contain a chosen fixed element of S.**

Corollary

Proof of the Proposition – continue:

The elements of type 3*3*4 in S10 are covered by 84 groups from MS3, and this is a minimal covering. In particular,

M=*MS3*-*D*, where

$D = \{H(0, k_1, k_2); k_1, k_2 \in \{1, 2, \ldots 9\}, k_2 < k_3\}$

is a minimal covering.

Proof:

According to the Theorem (*n*=10, *k*=3), the maximal subset $\{u_{1,} u_{2,} \ldots u_{t\}}$ of *U* with pairwise non-trivial intersection has cardinality: *m*=$\binom{9}{2}$=36. Therefore,

- 120-36=84.

Proof of Theorem 1

- We shall see that б(S_{10}) = |MS1|+|MS5|+|MS7|+84= 221 .
- The elements of order 21 are only to be found in MS1 and MS3, in both they are partitioned, so we take MS1={A_{10}}, size 1.
- The elements of order 10 are partitioned in MS6, MS7, and MS8. MS7 has the least size: 126.
- The elements of order 8 can be covered by MS4, or MS6, where they are partitioned, or by MS5 (that is the best choice), where they are not partitioned. However, not all 10 members S_9 from MS5 are needed: if we remove the one that fixes *i*, then a 8-cycle having 2 fixed points is still covered by the remaining subgroups of MS5. So 9 are enough.
- However, for the elements of order 14, type 2*7 we know the following: they are partitioned in MS3, and MS5. By removing H(0,k_1,k_2) from MS3, the elements (0,k_1)c_7, (o,k_2)c_7 and (k_1,k_2)c_7, where c_7 is an independent from 0, k_1, and k_2 cycle, are no longer covered. They can only be restored by adding 3 S_9-s – those that fix 0, k_1, and k_2 respectively. So, all 10 members of MS5 are needed.
- Together with the result for the elements of type 3*3*4, we have:
- б(S_{10})= 1+ 126+ 10+ 84 = 221.

Thank you!

QUESTIONS?