Beauville Groups

Emilio Pierro
Birkbeck, University of London

Groups St. Andrews, 3rd-11th August 2013

All groups will be finite!!!

Definitions

Definition. A Beauville surface $S = C_1 \times C_2/G$ is the quotient of the product of two complex projective varieties by a group, G, where G is a group of automorphisms of both C_1 and C_2 and acts freely on their product.

Definitions

Definition. A Beauville surface $S = C_1 \times C_2/G$ is the quotient of the product of two complex projective varieties by a group, G, where G is a group of automorphisms of both C_1 and C_2 and acts freely on their product. In addition, we require the projections

$$\beta_i : C_i \to \mathbb{P}^1(\mathbb{C})$$

ramify over 3 points.

Definitions

Definition. A Beauville surface $S = C_1 \times C_2/G$ is the quotient of the product of two complex projective varieties by a group, G, where G is a group of automorphisms of both C_1 and C_2 and acts freely on their product. In addition, we require the projections

$$\beta_i : C_i \to \mathbb{P}^1(\mathbb{C})$$

ramify over 3 points.

Remark. A celebrated theorem of Belyĭ states that if such a projection exists, then C_i can be defined over the field of algebraic numbers, $\overline{\mathbb{Q}}$.

Why are we interested in them?

1. Since our curves C_1, C_2 are Belyĭ we can associate a regular dessin in the sense of Grothendieck's theory of dessins d'enfants to each of them.

Why are we interested in them?

1. Since our curves C_1, C_2 are Belyĭ we can associate a regular dessin in the sense of Grothendieck's theory of dessins d'enfants to each of them.
2. Since Beauville surfaces are defined over $\overline{\mathbb{Q}}$ we can observe their behaviour under the action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Why are we interested in them?

1. Since our curves C_1, C_2 are Belyĭ we can associate a regular dessin in the sense of Grothendieck's theory of dessins d'enfants to each of them.
2. Since Beauville surfaces are defined over $\overline{\mathbb{Q}}$ we can observe their behaviour under the action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

For much much more on Beauville surfaces, see Ben Fairbairn's article in the proceedings of this very conference.

Definitions

Definition. Let G be a finite group and for $x, y \in G$ let

$$\Sigma(x,y) = \bigcup_{i=1}^{|G|} \bigcup_{g \in G} \{(x^i)^g, (y^i)^g, ((xy)^i)^g\}$$

Definitions

Definition. Let G be a finite group and for $x, y \in G$ let

$$\Sigma(x,y) = \bigcup_{i=1}^{|G|} \bigcup_{g \in G} \{(x^i)^g, (y^i)^g, ((xy)^i)^g\}$$

We say an unmixed Beauville structure on G is a quadruple $\{x_1, y_1, x_2, y_2\}$ such that $\langle x_1, y_1 \rangle = \langle x_2, y_2 \rangle = G$

Definitions

Definition. Let G be a finite group and for $x, y \in G$ let

$$\Sigma(x,y) = \bigcup_{i=1}^{|G|} \bigcup_{g \in G} \{(x^i)^g, (y^i)^g, ((xy)^i)^g\}$$

We say an unmixed Beauville structure on G is a quadruple $\{x_1, y_1, x_2, y_2\}$ such that $\langle x_1, y_1 \rangle = \langle x_2, y_2 \rangle = G$ and

$$\Sigma(x_1, y_1) \cap \Sigma(x_2, y_2) = \{1\}.$$

Definitions

Definition. Let G be a finite group and for $x, y \in G$ let

$$\Sigma(x,y) = \bigcup_{i=1}^{|G|} \bigcup_{g \in G} \{(x^i)^g, (y^i)^g, ((xy)^i)^g\}$$

We say an unmixed Beauville structure on G is a quadruple $\{x_1, y_1, x_2, y_2\}$ such that $\langle x_1, y_1 \rangle = \langle x_2, y_2 \rangle = G$ and

$$\Sigma(x_1, y_1) \cap \Sigma(x_2, y_2) = \{1\}.$$

If there exists a Beauville structure on a group, G, then we say that G is *Beauville.*

Definitions

Definition. Let G be a finite group and for $x, y \in G$ let

$$\Sigma(x,y) = \bigcup_{i=1}^{|G|} \bigcup_{g \in G} \{(x^i)^g, (y^i)^g, ((xy)^i)^g\}$$

We say an unmixed Beauville structure on G is a quadruple $\{x_1, y_1, x_2, y_2\}$ such that $\langle x_1, y_1 \rangle = \langle x_2, y_2 \rangle = G$ and

$$\Sigma(x_1, y_1) \cap \Sigma(x_2, y_2) = \{1\}.$$

If there exists a Beauville structure on a group, G, then we say that G is *Beauville.* The *type* of a Beauville Group will be $(l_1, m_1, n_1; l_2, m_2, n_2)$ where $l_i = o(x_i), m_i = o(y_i), n_i = o(x_i y_i)$.

Definitions

Definition. Let G be a finite group and for $x, y \in G$ let

$$\Sigma(x,y) = \bigcup_{i=1}^{|G|} \bigcup_{g \in G} \{(x^i)^g, (y^i)^g, ((xy)^i)^g\}$$

We say an unmixed Beauville structure on G is a quadruple $\{x_1, y_1, x_2, y_2\}$ such that $\langle x_1, y_1 \rangle = \langle x_2, y_2 \rangle = G$ and

$$\Sigma(x_1, y_1) \cap \Sigma(x_2, y_2) = \{1\}.$$

If there exists a Beauville structure on a group, G, then we say that G is *Beauville*. The *type* of a Beauville Group will be $(l_1, m_1, n_1; l_2, m_2, n_2)$ where $l_i = o(x_i), m_i = o(y_i), n_i = o(x_i y_i)$.

Remark. From the type of the Beauville structure we can compute a number of topological invariants associated to our Beauville Surface, $S = C_1 \times C_2 / G$.

Definitions

For instance:

1. From the Riemann-Hurwitz formula we can compute the genera g_1, g_2 of the curves C_1, C_2:

$$g_i = 1 + \frac{|G|}{2}\left(1 - \left(\frac{1}{o(x_i)} + \frac{1}{o(y_i)} + \frac{1}{o(x_i y_i)}\right)\right).$$

Definitions

For instance:

1. From the Riemann-Hurwitz formula we can compute the genera g_1, g_2 of the curves C_1, C_2:

$$g_i = 1 + \frac{|G|}{2}\left(1 - \left(\frac{1}{o(x_i)} + \frac{1}{o(y_i)} + \frac{1}{o(x_i y_i)}\right)\right).$$

2. Since the irregularity, q, of a Beauville surface is 0 we can compute the geometric genus, p_g:

$$p_g(S) = \frac{(g_1 - 1)(g_2 - 1)}{|G|} + q - 1.$$

Definitions

For instance:

1. From the Riemann-Hurwitz formula we can compute the genera g_1, g_2 of the curves C_1, C_2:

$$g_i = 1 + \frac{|G|}{2}\left(1 - \left(\frac{1}{o(x_i)} + \frac{1}{o(y_i)} + \frac{1}{o(x_i y_i)}\right)\right).$$

2. Since the irregularity, q, of a Beauville surface is 0 we can compute the geometric genus, p_g:

$$p_g(S) = \frac{(g_1 - 1)(g_2 - 1)}{|G|} + q - 1.$$

3. The Euler-Poincaré characteristic: $\chi = p_g - q + 1$.

Definitions

For instance:

1. From the Riemann-Hurwitz formula we can compute the genera g_1, g_2 of the curves C_1, C_2:

$$g_i = 1 + \frac{|G|}{2}\left(1 - \left(\frac{1}{o(x_i)} + \frac{1}{o(y_i)} + \frac{1}{o(x_i y_i)}\right)\right).$$

2. Since the irregularity, q, of a Beauville surface is 0 we can compute the geometric genus, p_g:

$$p_g(S) = \frac{(g_1 - 1)(g_2 - 1)}{|G|} + q - 1.$$

3. The Euler-Poincaré characteristic: $\chi = p_g - q + 1$.

4. From the Zeuthen-Segre Theorem we then get the Euler number:

$$e(S) = 4\frac{(g_1 - 1)(g_2 - 1)}{|G|}.$$

Definitions

Definition. (Fuertes, González-Diez, Jaikin-Zapirain, 2011) Let G be a finite group. By the *Beauville genus spectrum* of G we mean the set $\mathrm{Spec}(G)$ of pairs of integers (g_1, g_2) such that $g_1 \leq g_2$ and there are curves C_1, C_2 of genera g_1 and g_2 with an action of G on $C_1 \times C_2$ such that $S = C_1 \times C_2/G$ is a Beauville surface.

Definitions

Definition. (Fuertes, González-Diez, Jaikin-Zapirain, 2011) Let G be a finite group. By the *Beauville genus spectrum* of G we mean the set $\mathrm{Spec}(G)$ of pairs of integers (g_1, g_2) such that $g_1 \leq g_2$ and there are curves C_1, C_2 of genera g_1 and g_2 with an action of G on $C_1 \times C_2$ such that $S = C_1 \times C_2/G$ is a Beauville surface.

Theorem (Fuertes, González-Diez, Jaikin-Zapirain, 2011)

1. $\mathrm{Spec}(S_5) = \{(19, 21)\}$.
2. $\mathrm{Spec}(PSL_2(7)) = \{(8, 49), (15, 49), (17, 22), (22, 33), (22, 49)\}$.
3. For $(n, 6) = 1$,

$$\mathrm{Spec}(\mathbb{Z}/n\mathbb{Z})^2 = \left\{ \left(\frac{(n-1)(n-2)}{2}, \frac{(n-1)(n-2)}{2} \right) \right\}.$$

Definitions

Proposition (ep, GAP, 2013)

The genus spectrum for the following groups has been computed in:
$PGL_2(7)$, $PGL_2(9)$, $PSL_2(8)$,
$PSU_3(3):2$, $P\Sigma L_2(25)$, $P\Sigma L_2(2,7)$,
$PSL_3(3)$, $PSL_3(3).2$, $PSL_3(4)$,
$PSU_3(3)$, $PSU_3(4)$, $PSU_4(2)$,
A_6, A_7, A_8, A_9, A_{10},
S_6, S_7, S_8, S_9, S_{10},
M_{11}, M_{12}, M_{22}, M_{23},
$Sz(8)$, J_1, J_2, ...

Examples of Beauville Groups

What groups are known to be Beauville?

Examples of Beauville Groups

What groups are known to be Beauville?

Theorem (Catanese, 2000)

The only finite abelian Beauville groups are the groups $G = \mathbb{Z}_n \oplus \mathbb{Z}_n$ *where* $n > 1$ *and* $(n, 6) = 1$.

Examples of Beauville Groups

What groups are known to be Beauville?

Theorem (Catanese, 2000)

The only finite abelian Beauville groups are the groups $G = \mathbb{Z}_n \oplus \mathbb{Z}_n$ *where* $n > 1$ *and* $(n, 6) = 1$.

Conjecture (Catanese, 2005)

With the exception of A_5*, all finite non-abelian simple groups are Beauville.*

Examples of Beauville Groups

Theorem (Garion, Larsen, Lubotzky; Guralnick, Malle 2012)

Let G be a finite non-abelian simple group other than A_5. Then G admits an unmixed Beauville structure.

Examples of Beauville Groups

Theorem (Garion, Larsen, Lubotzky; Guralnick, Malle 2012)

Let G be a finite non-abelian simple group other than A_5. Then G admits an unmixed Beauville structure.

Theorem (Fairbairn, Magaard, Parker, 2012)

With the exceptions of A_5 and $SL_2(5)$, every finite quasisimple group is a Beauville group.

Examples of Beauville Groups

Theorem (Garion, Larsen, Lubotzky; Guralnick, Malle 2012)

Let G be a finite non-abelian simple group other than A_5. Then G admits an unmixed Beauville structure.

Theorem (Fairbairn, Magaard, Parker, 2012)

With the exceptions of A_5 and $SL_2(5)$, every finite quasisimple group is a Beauville group.

Remark. Certain families of simple groups had elsewhere been shown to be Beauville e.g. Fuertes, González-Diez (2010) had shown the alternating groups A_n for $n > 5$ and Fuertes, Jones (2011) had shown the Suzuki groups and Small Ree groups are Beauville.

More Examples of Beauville Groups

What else do we know?

More Examples of Beauville Groups

What else do we know? What about the almost simple groups?

More Examples of Beauville Groups

What else do we know? What about the almost simple groups?

Definition. Let G be a finite group. We say G is *almost simple* if there exists a finite simple group, H, such that $H \leqslant G \leqslant \mathrm{Aut}(H)$.

More Examples of Beauville Groups

Theorem (Fuertes, González-Diez, 2010)

Let $n \geq 5$. Then S_n is Beauville.

More Examples of Beauville Groups

Theorem (Fuertes, González-Diez, 2010)

Let $n \geq 5$. Then S_n is Beauville.

Theorem (Garion, 2010)

Let $q = p^e$ be a prime power where $q \geq 7$. Then $PGL_2(q)$ is Beauville.

More Examples of Beauville Groups

Theorem (Fuertes, González-Diez, 2010)

Let $n \geq 5$. Then S_n is Beauville.

Theorem (Garion, 2010)

Let $q = p^e$ be a prime power where $q \geq 7$. Then $PGL_2(q)$ is Beauville.

Proposition (ep, 2013)

Let G be a sporadic group whose outer automorphism group is non-trivial. Then $Aut(G)$ is Beauville.

Examples of non-Beauville Groups

Definition. The socle of G, $soc(G)$, is the subgroup of G generated by the minimal normal subgroups of G.

Examples of non-Beauville Groups

Definition. The socle of G, $soc(G)$, is the subgroup of G generated by the minimal normal subgroups of G.

Observation. For $x, y \in G$, if $\langle x, y, xy \rangle = G$ then without loss of generality, $x, y \in G \setminus soc(G)$.

Examples of non-Beauville Groups

Recall the following:

Examples of non-Beauville Groups

Recall the following:

1. The Suzuki groups $Sz(2^{2n+1})$ do not contain elements of order 3.

Examples of non-Beauville Groups

Recall the following:

1. The Suzuki groups $Sz(2^{2n+1})$ do not contain elements of order 3.
2. Their only outer automorphisms come from the Frobenius automorphims of the underlying field of order 2^{2n+1}.

Examples of non-Beauville Groups

Recall the following:

1. The Suzuki groups $Sz(2^{2n+1})$ do not contain elements of order 3.
2. Their only outer automorphisms come from the Frobenius automorphims of the underlying field of order 2^{2n+1}.

Now consider the case when $3 \,|\, 2n+1$.

Examples of non-Beauville Groups

Recall the following:

1. The Suzuki groups $Sz(2^{2n+1})$ do not contain elements of order 3.
2. Their only outer automorphisms come from the Frobenius automorphims of the underlying field of order 2^{2n+1}.

Now consider the case when $3 \mid 2n+1$. We have the following:

Proposition

Let G be the automorphism group of the Suzuki group $Sz(8^{2n+1})$. Then G is not Beauville.

Examples of non-Beauville Groups

Recall the following:

1. The Suzuki groups $Sz(2^{2n+1})$ do not contain elements of order 3.
2. Their only outer automorphisms come from the Frobenius automorphims of the underlying field of order 2^{2n+1}.

Now consider the case when $3 \mid 2n+1$. We have the following:

Proposition

Let G be the automorphism group of the Suzuki group $Sz(8^{2n+1})$. Then G is not Beauville.

Remark. In particular this argument will also apply to the groups $\text{Aut}(PSL_2(q)) = P\Gamma L_2(q)$ where $q = p^e \geq 7$ is a prime power and e is coprime to $q, q-1$ and $q+1$.

Examples of non-Beauville Groups

Recall the outer automorphisms of the group $G = PSL_2(9)(\cong A_6)$.

Examples of non-Beauville Groups

Recall the outer automorphisms of the group $G = PSL_2(9)(\cong A_6)$.

1. The diagonal outer automorphism yields $PGL_2(9)$ which is Beauville.

Examples of non-Beauville Groups

Recall the outer automorphisms of the group $G = PSL_2(9)(\cong A_6)$.

1. The diagonal outer automorphism yields $PGL_2(9)$ which is Beauville.
2. The field outer automorphism yields $P\Sigma L_2(9)(\cong S_6)$ which is Beauville.

Examples of non-Beauville Groups

Recall the outer automorphisms of the group $G = PSL_2(9)(\cong A_6)$.

1. The diagonal outer automorphism yields $PGL_2(9)$ which is Beauville.
2. The field outer automorphism yields $P\Sigma L_2(9)(\cong S_6)$ which is Beauville.
3. The other outer automorphism yields the Mathieu group M_{10} which is *not* Beauville.

Examples of non-Beauville Groups

Recall the outer automorphisms of the group $G = PSL_2(9)(\cong A_6)$.

1. The diagonal outer automorphism yields $PGL_2(9)$ which is Beauville.
2. The field outer automorphism yields $P\Sigma L_2(9)(\cong S_6)$ which is Beauville.
3. The other outer automorphism yields the Mathieu group M_{10} which is *not* Beauville. Why?

Examples of non-Beauville Groups

Recall the outer automorphisms of the group $G = PSL_2(9)(\cong A_6)$.

1. The diagonal outer automorphism yields $PGL_2(9)$ which is Beauville.
2. The field outer automorphism yields $P\Sigma L_2(9)(\cong S_6)$ which is Beauville.
3. The other outer automorphism yields the Mathieu group M_{10} which is *not* Beauville. Why?

How does this affect the full automorphism group $P\Gamma L_2(9)$?

Examples of non-Beauville Groups

Recall the outer automorphisms of the group $G = PSL_2(9)(\cong A_6)$.

1. The diagonal outer automorphism yields $PGL_2(9)$ which is Beauville.
2. The field outer automorphism yields $P\Sigma L_2(9)(\cong S_6)$ which is Beauville.
3. The other outer automorphism yields the Mathieu group M_{10} which is *not* Beauville. Why?

How does this affect the full automorphism group $P\Gamma L_2(9)$?

What about $PSL_2(p^2)$ in general?

Examples of non-Beauville Groups

Recall the outer automorphisms of the group $G = PSL_2(9)(\cong A_6)$.

1. The diagonal outer automorphism yields $PGL_2(9)$ which is Beauville.
2. The field outer automorphism yields $P\Sigma L_2(9)(\cong S_6)$ which is Beauville.
3. The other outer automorphism yields the Mathieu group M_{10} which is *not* Beauville. Why?

How does this affect the full automorphism group $P\Gamma L_2(9)$?

What about $PSL_2(p^2)$ in general?

Proposition

Let $H = PSL_2(p^2)$ for $p > 2$ prime. Then both $H.2$ the non-split extension by an outer automorphism and $Aut(H)$ the full automorphism group of H are not Beauville.

Thank you for your time.