Geometric actions of classical groups

Raffaele Rainone

School of Mathematics
University of Southampton

Groups St. Andrews
August 2013

Algebraic groups

Let k be an algebraically closed field of characteristic $p \geqslant 0$.

An algebraic group G is an affine algebraic variety, defined over k, with a group structure such that

$$\begin{array}{rccc} \mu: & G \times G & \to & G \\ & (x, y) & \mapsto & xy \end{array} \qquad \begin{array}{rccc} \iota: & G & \to & G \\ & x & \mapsto & x^{-1} \end{array}$$

are morphisms of varieties.

Example

The prototype is the **special linear group**

$$\mathrm{SL}_n(k) = \{A \in \mathcal{M}_n(k) \mid \det(A) = 1\}$$

Actions of algebraic groups

Let G be an algebraic group and Ω a variety (over k). An action of G on Ω is a morphism of varieties (with the usual properties)

$$\begin{array}{ccc} G \times \Omega & \to & \Omega \\ (x, \omega) & \mapsto & x.\omega \end{array}$$

We can define orbits and stabilisers as usual:

- orbits are locally closed subsets of Ω, and we can define

$$\dim G.x = \dim \overline{G.x}$$

- for $\omega \in \Omega$, $G_\omega \leqslant G$ is closed

Actions of algebraic groups

Let G be an algebraic group and Ω a variety (over k). An action of G on Ω is a morphism of varieties (with the usual properties)

$$\begin{array}{ccc} G \times \Omega & \to & \Omega \\ (x, \omega) & \mapsto & x.\omega \end{array}$$

We can define orbits and stabilisers as usual:

- orbits are locally closed subsets of Ω, and we can define

$$\dim G.x = \dim \overline{G.x}$$

- for $\omega \in \Omega$, $G_\omega \leqslant G$ is closed

Lemma

Let $H \leqslant G$ be a closed subgroup. Then

(i) *G/H is a (quasi-projective) variety*

(ii) *there is a natural (transitive) action $G \times G/H \to G/H$*

Fixed point spaces

Let G be an algebraic group acting on a variety Ω. For any $x \in G$, the fixed point space

$$C_\Omega(x) = \{\omega \in \Omega : x.\omega = \omega\} \subseteq \Omega$$

is closed.

Proposition

Let $\Omega = G/H$. Then, for $x \in G$,

$$\dim C_\Omega(x) = \begin{cases} 0 & \text{if } x^G \cap H = \emptyset \\ \dim \Omega - \dim x^G + \dim(x^G \cap H) & \text{otherwise} \end{cases}$$

General aim: given $x \in G$ of prime order, derive bounds on

$$f_\Omega(x) = \frac{\dim C_\Omega(x)}{\dim \Omega}$$

Classical groups

Let V be an n-dimensional k-vector space.

$$\begin{aligned} \mathrm{GL}(V) &= \text{invertible linear maps } V \to V \\ \mathrm{Sp}(V) &= \{x \in \mathrm{GL}(V) : \beta(x.u, x.v) = \beta(u, v)\} \\ \mathrm{O}(V) &= \{x \in \mathrm{GL}(V) : Q(x.u) = Q(u)\} \end{aligned}$$

where: β is a symplectic form on V
Q is a non-degenerate quadratic form on V.

We write $Cl(V)$ for $\mathrm{GL}(V), \mathrm{Sp}(V), \mathrm{O}(V)$
Similarly Cl_n for $\mathrm{GL}_n, \mathrm{Sp}_n, \mathrm{O}_n$

Let $G = Cl(V)$ be a classical group.

We define 5 families of positive-dimension subgroups that arise naturally from the underlying geometry of V

- $\mathscr{C}_1$ stabilisers of subspaces $U \subset V$
- $\mathscr{C}_2$ stabilisers of direct sum decompositions
$$V = V_1 \oplus \ldots \oplus V_t$$
- $\mathscr{C}_3$ stabilisers of totally singular decompositions $V = U \oplus W$, when $G = \mathrm{Sp}(V)$ or $\mathrm{O}(V)$
- $\mathscr{C}_4$ stabilisers of tensor product decompositions
$$V = V_1 \otimes \ldots \otimes V_t$$
- $\mathscr{C}_5$ stabiliser of non-degenerate forms on V

Set $\mathscr{C}(G) = \bigcup \mathscr{C}_i$.

Subgroup structure

Example

$\mathscr{C}_2$ Let $G = \mathrm{GL}_n$. Assume $V = V_1 \oplus \ldots \oplus V_t$ where $\dim V_i = n/t$. Then $H = \mathrm{GL}_{n/t} \wr S_t$, and

$$H^\circ = \mathrm{GL}_{n/t} \times \ldots \times \mathrm{GL}_{n/t}$$

$\mathscr{C}_3$ Let $G = \mathrm{Sp}_n$. Assume $V = U \oplus W$ where U, W are maximal totally singular subspaces. Then $H = \mathrm{GL}_{n/2}.2$ and

$$H^\circ = \left\{ \begin{pmatrix} A & \\ & A^{-t} \end{pmatrix} : A \in \mathrm{GL}_{n/2} \right\} \cong \mathrm{GL}_{n/2}$$

Subgroup structure

Example

$\mathscr{C}_2$ Let $G = \mathrm{GL}_n$. Assume $V = V_1 \oplus \ldots \oplus V_t$ where $\dim V_i = n/t$. Then $H = \mathrm{GL}_{n/t} \wr S_t$, and

$$H^\circ = \mathrm{GL}_{n/t} \times \ldots \times \mathrm{GL}_{n/t}$$

$\mathscr{C}_3$ Let $G = \mathrm{Sp}_n$. Assume $V = U \oplus W$ where U, W are maximal totally singular subspaces. Then $H = \mathrm{GL}_{n/2}.2$ and

$$H^\circ = \left\{ \begin{pmatrix} A & \\ & A^{-t} \end{pmatrix} : A \in \mathrm{GL}_{n/2} \right\} \cong \mathrm{GL}_{n/2}$$

Theorem (Liebeck - Seitz, 1998)

Let $G = SL(V), Sp(V)$ or $SO(V)$ and $H \leqslant G$ closed and positive dimensional. Then either H is contained in a member of $\mathscr{C}(G)$, or H° is simple and acts irreducibly on V.

Aim

$G = Cl(V)$ classical algebraic group
$H \leqslant G$ closed geometric subgroup
$\Omega = G/H$

Main aim

Derive bounds on

$$f_\Omega(x) = \frac{\dim C_\Omega(x)}{\dim \Omega}$$

for all $x \in G$ of prime order.

Further aims

- sharpness, characterisazions?
- "Local bounds": how does the action of x on V influence $f_\Omega(x)$?

Background

Let G be a simple algebraic group, $H \leqslant G$ closed. Set $\Omega = G/H$.

Theorem (Lawther, Liebeck, Seitz (2002))

If G exceptional then, for $x \in G$ of prime order,

$$f_\Omega(x) \leqslant \delta(G, H, x)$$

Theorem (Burness, 2003)

Either there exists an involution $x \in G$

$$f_\Omega(x) = \frac{\dim C_\Omega(x)}{\dim \Omega} \geqslant \frac{1}{2} - \epsilon$$

for a small $\epsilon \geqslant 0$, or (G, Ω) is in a short list of known exceptions.

Background

Further motivation arises from finite permutation group.

Let Ω be a finite set and $G \leqslant \mathrm{Sym}(\Omega)$. For $x \in G$, the fixed point ratio is defined

$$\mathrm{fpr}_{\Omega}(x) = \frac{|C_{\Omega}(x)|}{|\Omega|}$$

If G is transitive with point stabiliser H then

$$\mathrm{fpr}_{\Omega}(x) = \frac{|x^G \cap H|}{|x^G|}$$

Background

Further motivation arises from finite permutation group.

Let Ω be a finite set and $G \leqslant \mathrm{Sym}(\Omega)$. For $x \in G$, the fixed point ratio is defined

$$\mathrm{fpr}_\Omega(x) = \frac{|C_\Omega(x)|}{|\Omega|}$$

If G is transitive with point stabiliser H then

$$\mathrm{fpr}_\Omega(x) = \frac{|x^G \cap H|}{|x^G|}$$

Bounds on fpr have been studied and applied to a variety of problems, e.g.

- base sizes
- monodromy groups of covering of Riemann surfaces
- (random) generation of simple groups

Fixed point spaces

Let $G = Cl(V)$, $H \leqslant G$ closed and $\Omega = G/H$.

Recall, for $x \in H$ fixed,

$$\dim C_\Omega(x) = \dim \Omega - \dim x^G + \dim(x^G \cap H)$$

To compute $\dim C_\Omega(x)$ we need:

(i) information on the centraliser $C_G(x)$, so

$$\dim x^G = \dim G - \dim C_G(x)$$

(ii) informations on the fusion of H-classes in G, so we can compute $\dim(x^G \cap H)$.

Conjugacy classes I

For $x \in \mathrm{GL}_n$ we have $x = x_s x_u = x_u x_s$.

Conjugacy classes I

For $x \in \mathrm{GL}_n$ we have $x = x_s x_u = x_u x_s$. Up to conjugation,

$$x_s = [\lambda_1 I_{a_1}, \lambda_2 I_{a_2}, \ldots, \lambda_n I_{a_n}], \quad x_u = [J_n^{a_n}, \ldots, J_1^{a_1}]$$

Conjugacy classes I

For $x \in \mathrm{GL}_n$ we have $x = x_s x_u = x_u x_s$. Up to conjugation,

$$x_s = [\lambda_1 I_{a_1}, \lambda_2 I_{a_2}, \ldots, \lambda_n I_{a_n}], \quad x_u = [J_n^{a_n}, \ldots, J_1^{a_1}]$$

Fact

Let s, s' and u, u' in $G = Cl(V)$. Then $s \sim_G s'$, $u \sim_G u'$ if, and only if, they are $\mathrm{GL}(V)$-conjugate (unless $p = 2$ and u, u' are unipotent).

Conjugacy classes I

For $x \in \mathrm{GL}_n$ we have $x = x_s x_u = x_u x_s$. Up to conjugation,

$$x_s = [\lambda_1 I_{a_1}, \lambda_2 I_{a_2}, \ldots, \lambda_n I_{a_n}], \quad x_u = [J_n^{a_n}, \ldots, J_1^{a_1}]$$

Fact

Let s, s' and u, u' in $G = Cl(V)$. Then $s \sim_G s'$, $u \sim_G u'$ if, and only if, they are $\mathrm{GL}(V)$-conjugate (unless $p = 2$ and u, u' are unipotent).

It is well known how to compute $\dim x^G$ for unipotent and semisimple elements. For example if $G = \mathrm{GL}_n$:

$$\dim x_s^G = n^2 - \sum_{i=1}^{n} a_i^2$$

$$\dim x_u^G = n^2 - 2 \sum_{1 \le i < j \le n} i a_i a_j - \sum_{i=1}^{n} i a_i^2$$

Conjugacy classes II

Recall: $\dim C_\Omega(x) = \dim \Omega - \dim x^G + \dim(x^G \cap H)$.

Conjugacy classes II

Recall: $\dim C_\Omega(x) = \dim \Omega - \dim x^G + \dim(x^G \cap H)$.

In general it is hard to compute $\dim(x^G \cap H)$, but the following result is useful:

Theorem (Guralnick, 2007)

If H° is reductive then

$$x^G \cap H = x_1^H \cup \ldots \cup x_m^H$$

for some m. Thus $\dim(x^G \cap H) = \max_i\{\dim x_i^H\}$.

Example

Let $G = \mathrm{GL}_{18}$, $H = \mathrm{GL}_6 \wr S_3$ and $p = 3$. Set $\Omega = G/H$, thus $\dim \Omega = 18^2 - 3 \cdot 6^2 = 216$. Let

$$x = [J_3^2, J_2^3, J_1^6], \ \dim x^G = 174$$

Then $x^G \cap H = x^G \cap H^\circ$ and $x^G \cap H = \bigcup_{i=1}^4 x_i^H$ where

$$x_1 = [J_3^2 \mid J_2^2, J_1^2 \mid J_2, J_1^4], \ \ x_2 = [J_3^2 \mid J_2^3 \mid J_1^6],$$

$$x_3 = [J_3, J_2, J_1 \mid J_3, J_2, J_1 \mid J_2, J_1^4], \ \ x_4 = [J_3, J_2, J_1 \mid J_3, J_1^3 \mid J_2^2, J_1^2]$$

and

$$\dim x_1^H = 46, \ \dim x_2^H = 42, \ \dim x_3^H = 54, \ \dim x_4^H = 52$$

Thus $\dim(x^G \cap H) = 54$. Therefore $f_\Omega(x) = \frac{4}{9} > \frac{1}{3}$.

Main result: Global bounds

Recall that $H \in \mathscr{C}_2 \cup \mathscr{C}_3$ is a stabiliser of a decomposition $V = V_1 \oplus \ldots \oplus V_t$.

Theorem (R., 2013)

Let $G = Cl_n$ and $H \in \mathscr{C}_2 \cup \mathscr{C}_3$. Set $\Omega = G/H$ and fix $x \in H$ of prime order r. Then

$$\frac{1}{r} - \epsilon \leqslant f_\Omega(x) = \frac{\dim C_\Omega(x)}{\dim \Omega} \leqslant 1 - \frac{1}{n}$$

where

$$\epsilon = \begin{cases} 0 & r = p \\ \frac{1}{r} & p \neq r > n \\ \frac{rt^2}{4n^2(t-1)} & p \neq r \leqslant n \end{cases}$$

Main result: Global bounds

Let $G = Cl(V)$ and $H \in \mathscr{C}_2 \cup \mathscr{C}_3$. Set $\Omega = G/H$.

Let $M = \max_{x \in G \setminus Z(G)} \{f_\Omega(x)\}$.

Theorem (R., 2013)

Let r be a prime. Then either

(i) *there exists $x \in G$ of order r such that $f_\Omega(x) = M$; or,*

(ii) *(G, H) belong to a short list of known exceptions.*

For example, if $G = \mathrm{GL}_n$ and $H = \mathrm{GL}_1 \wr S_n$ then

$$f_\Omega(x) = M = 1 - \frac{2}{n} + \frac{1}{n(n-1)}$$

if, and only if, $x = [1, -I_{n-1}]$ or $[J_2, J_1^{n-2}]$.

Main result: Local bounds

For $x \in Cl(V)$, we define $\nu(x)$ to be the co-dimension of the largest eigenspace of x. For example, if

$$x = [J_n^{a_n}, \ldots, J_1^{a_1}]$$

then $\nu(x) = n - \sum_{i=1}^{n} a_i$.

Main result: Local bounds

For $x \in Cl(V)$, we define $\nu(x)$ to be the co-dimension of the largest eigenspace of x. For example, if

$$x = [J_n^{a_n}, \ldots, J_1^{a_1}]$$

then $\nu(x) = n - \sum_{i=1}^{n} a_i$.

Theorem (R., 2013)

Let $G = Cl(V)$ and $H \in \mathscr{C}_2 \cup \mathscr{C}_3$. Let $x \in G$ of prime order r, with $\nu(x) = s$. Then

$$1 - \frac{s(2n-s)}{n(n-2)} - \frac{4}{n} \leqslant f_\Omega(x) \leqslant 1 - \frac{s}{n} + \frac{1}{n}$$

1. Let $G = Cl(V)$ and $H \in \mathscr{C}_2$.

(i) Let $x \in H^{\circ}$ be unipotent. Can we find an explicit formula for $\dim(x^G \cap H^{\circ})$?

(ii) Derive local lower bounds on $f_{\Omega}(x)$ for $x \in G$ unipotent with $\nu(x) = s$.

(iii) Can we give an exact formula for $f_{\Omega}(x)$ when $x \in G$ is an involution?

2. Same analysis for $\mathscr{C}_4$ subgroups (stabilisers of $V = V_1 \otimes V_2$ or $V = \bigotimes_{i=1}^{t} V_i$).

3. Explore applications (e.g. derive bounds on fpr's for finite groups of Lie type, etc).

THANK YOU!