Recent Results on Generalized Baumslag-Solitar Groups

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(i) A *Baumslag-Solitar group* is a group with a presentation

$$BS(m,n) = < t, x \mid (x^m)^t = x^n >,$$

where $m, n \in \mathbb{Z}^* = \mathbb{Z} \backslash \{0\}$.

(ii) A similar type of 1-relator group is

$$K(m,n) = < x, y \mid x^m = y^n >,$$

where $m, n \in \mathbb{Z}^*$.

These are the fundamental groups of certain graphs of groups.

GBS-graphs

Let Γ be a finite connected graph. For each edge e label the endpoints e^+ and e^-. Infinite cyclic groups $\langle g_x \rangle$ and $\langle u_e \rangle$ are assigned to each vertex x and edge e.

Injective homomorphisms $\langle u_e \rangle \to \langle g_{e^+} \rangle$ and $\langle u_e \rangle \to \langle g_{e^-} \rangle$ are defined by

$$u_e \mapsto g_{e^+}^{\omega^+(e)} \text{ and } \; u_e \mapsto g_{e^-}^{\omega^-(e)}$$

where $\omega^+(e),\ \omega^-(e) \in \mathbb{Z}^*$.

So we have a *weight function*

$$\omega : E(\Gamma) \to \mathbb{Z}^* \times \mathbb{Z}^*$$

where $\omega(e) = (\omega^-(e), \omega^+(e))$ is defined up to $\pm$. The weighted graph

$$(\Gamma, \omega)$$

is a *generalized Baumslag-Solitar graph* or *GBS-graph*.

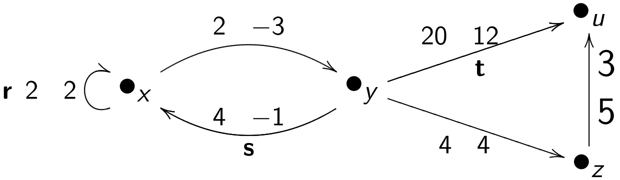
GBS-groups

The *generalized Baumslag-Solitar group* (GBS-group) determined by the GBS-graph (Γ, ω) is the fundamental group $G = \pi_1(\Gamma, \omega)$. If T is a maximal subtree of Γ, then G has generators g_x and t_e, with relations

$$\begin{cases} g_{e^+}^{\omega^+(e)} & = \; g_{e^-}^{\omega^-(e)}, \text{ for } e \in E(T), \\ \\ (g_{e^+}^{\omega^+(e)})^{t_e} & = \; g_{e^-}^{\omega^-(e)}, \text{ for } e \in E(\Gamma) \backslash E(T). \end{cases}$$

If Γ is an edge e, $G = K(m,n)$: if Γ is a single loop e, $G = BS(m,n)$, where $m = \omega^+(e), n = \omega^-(e)$.

An example



The maximal subtree T is the path x, y, z, u. The GBS-group has a presentation in $r, s, t, g_x, g_y, g_z, g_u$ with relations

$$g_x^2 = g_y^{-3},\ g_y^4 = g_z^4,\ g_z^5 = g_u^3$$

$$(g_x^2)^r = g_x^2,\ (g_x^4)^s = g_y^{-1},\ (g_u^{12})^t = g_y^{20}.$$

Some properties of GBS-groups

Let $G = \pi_1(\Gamma, \omega)$ be a GBS-group.

(i) G is independent of the choice of maximal subtree.

(ii) *G is finitely presented and torsion-free.*

(iii) *If Γ is a tree, then G is residually finite and hence is hopfian.*

The next result is due to P. Kropholler.

(iv) *The non-cyclic GBS-groups are exactly the finitely generated groups of cohomological dimension 2 which have a commensurable infinite cyclic subgroup.*

Some properties of GBS-groups

(v) *If H is a finitely generated subgroup of a GBS-group G, either H is a GBS-group or it is free. Hence G is coherent.*

Proof. We have $cd(H) \leq cd(G) = 2$. If $cd(H) = 1$, then H is free by the Stallings-Swan Theorem. Otherwise $cd(H) = 2$. If H contains a commensurable element, it is a GBS-group by (iv). If H has no commensurable elements, it is free.

(vi) *The second derived subgroup of a GBS-group is free.* (Kropholler.)

The weight of a path

Let (Γ, ω) be a *GBS*-graph with a maximal subtree T. Let $e = \langle x, y \rangle$ be a non-tree edge where $x \neq y$. There is a unique path in T from x to y, say

$$x = x_0, x_1, \ldots, x_n = y.$$

Then there is a relation in $G = \pi_1(\Gamma, \omega)$

$$g_x^{p_1(e)} = g_y^{p_2(e)}$$

where $p_1(e)$ and $p_2(e)$ are the products of the left and right weight values of the edges in the tree path $[x, y]$.

Lemma 1. *Let* (Γ, ω) *be a GBS-graph with a maximal subtree* T*. Let* $\alpha = [x, y]$ *be a path in* T*. Then there exist* $a, b \in \mathbb{Z}^*$ *such that* $g_x^a = g_y^b$ *in* $\pi_1(\Gamma, \omega)$*. Also, if* $g_x^m = g_y^n$*, then* $(m, n) = (a, b)q$ *for some* $q \in \mathbb{Z}^*$*.*

Definition. Call (a, b) the *weight of the path* α in T and denote it by $\omega_T(\alpha)$ or

$$\omega_T(x, y) = (\omega_T^{(1)}(x, y), \omega_T^{(2)}(x, y)).$$

This is unique up to $\pm$.

How to compute the weight of a path

Let α be the path $x = x_0, x_1, \dots, x_n = y$ and write $\omega(\langle x_i, x_{i+1}\rangle) = (u_i^{(1)},\ u_i^{(2)}),\ i = 0, 1, \dots, n-1$. Define $(\ell_i, m_i),\ 0 \le i \le n$, recursively by $\ell_0 = 1 = m_0$ and

$$\ell_{i+1} = \frac{\ell_i u_i^{(1)}}{\gcd(m_i, u_i^{(1)})}, \qquad m_{i+1} = \frac{m_i u_i^{(2)}}{\gcd(m_i, u_i^{(1)})}.$$

Then

Lemma 2. $\omega_T(x, y) = (\ell_n, m_n)$.

Let (Γ, ω) be a GBS-graph with a maximal subtree T. The non-tree edge $e =< x, y >$ is called *T-dependent* or *skew T-dependent* if and only if

$$\frac{\omega^-(e)}{\omega^+(e)} = \frac{\omega_T^{(1)}(e)}{\omega_T^{(2)}(e)} \quad \text{or} \quad -\frac{\omega_T^{(1)}(e)}{\omega_T^{(2)}(e)}$$

respectively. If e is a loop, then e is T-dependent (skew T-dependent) if and only if $\omega^-(e) = \omega^+(e)$ or $\omega^-(e) = -\omega^+(e)$ respectively.

If every non-tree edge of a GBS-graph is T-dependent, the GBS-graph is called *tree dependent*.

If every non-tree edge is T-dependent or skew T-dependent with at least of the latter, then the GBS-graph is called *skew tree dependent*.

These properties are independent of the choice of T.

Tree dependence is relevant to the computation of homology in low dimensions.

Theorem 1. (DR). *Let $G = \pi_1(\Gamma, \omega)$ be a GBS-group. Then the torsion-free rank of $H_1(G) = G_{ab}$ is*

$$r_0(G) = |E(\Gamma)| - |V(\Gamma)| + 1 + \epsilon$$

where $\epsilon = 1$ if (Γ, ω) is tree dependent and otherwise $\epsilon = 0$. Hence tree dependence is independent of the choice of maximal subtree.

Theorem 2. (DR). *For any GBS-group G the Schur multiplier $H_2(G)$ is free abelian of rank $r_0(G) - 1$.*

The Δ-function

Let G be a group with a commensurable element x of infinite order. If $g \in G$, then $\langle x \rangle \cap \langle x \rangle^g \neq 1$ and $(x^n)^g = x^m$ for $m, n \in \mathbb{Z}^*$. Define $\Delta_x(g) = \frac{m}{n}$. Then

$$\Delta_x : G \mapsto \mathbb{Q}^*$$

is a well defined homomorphism.

If $y \in G$ is commensurable and $\langle x \rangle \cap \langle y \rangle \neq 1$, then $\Delta_x = \Delta_y$. If this holds for all commensurable elements, then Δ_x depends only on G: denote it by

$$\Delta^G.$$

The Δ-function of a GBS-group

A GBS-graph (Γ, ω) or the group $G = \pi_1(\Gamma, \omega)$, is called *elementary* if $G \simeq BS(1, \pm 1)$. If G is non-elementary, then each commensurable element of G is elliptic and hence is conjugate to a power of some g_v. Hence Δ^G is unique.

Lemma 3. *Let (Γ, ω) be a non-elementary GBS-graph, with T a maximal subtree, and let $G = \pi_1(\Gamma, \omega)$. Then:*

(i) $\Delta^G(g_v) = 1$ for all $v \in V(\Gamma)$;

(ii) If $e \in E(\Gamma) \backslash E(T)$, $\omega(e) = (a, b)$, $\omega_T(e) = (m, n)$,

$$\Delta^G(t_e) = \frac{an}{bm}.$$

Corollary. (G. Levitt). *Let e be a non-tree edge. Then:*

(i) e is T-dependent if and only if $\Delta^G(t_e) = 1$*. Hence* (Γ, ω) *is tree dependent if and only if* Δ^G *is trivial.*

(ii) e is skew T-dependent if and only if $\Delta^G(t_e) = -1$*. Hence* (Γ, ω) *is skew tree dependent if and only if* $\mathrm{Im}(\Delta^G) = \{\pm 1\}$.

If $\mathrm{Im}(\Delta^G) \subseteq \{\pm 1\}$, call $G = \pi_1(\Gamma, \omega)$ *unimodular*.

The centre of a GBS-group

The following result tells us when the center of a GBS-group is non-trivial.

Theorem 3. *Let* (Γ, ω) *be a GBS-graph and let* G *be its fundamental group. Assume that* G *is non-elementary. Then the following are equivalent.*

(a) $Z(G)$ *is non-trivial.*

(b) Δ^G *is trivial.*

(c) (Γ, ω) *is tree-dependent.*

Let (Γ, ω) be a GBS-graph. In finding $Z(G)$ we may assume the graph is non-elementary. We can also assume (Γ, ω) is tree dependent since otherwise $Z(G) = 1$.

In a GBS-graph the *distal weight* of a leaf in a maximal subtree is the weight occurring at the vertex of degree 1. In finding the centre there is no loss in assuming there are no leaves with distal weight ± 1.

Locating the centre

Lemma 4. *Let* (Γ, ω) *be a non-elementary GBS-graph with a maximal subtree* T*. Assume no leaves of* T *have distal weight* ± 1*. Then*

$$Z(G) \leq \bigcap_{x \in V(\Gamma)} \langle g_x \rangle.$$

For any $x, v \in V(\Gamma)$, $\langle g_x \rangle \cap \langle g_v \rangle = \langle g_v^{\omega_T^{(1)}(v,x)} \rangle$. Hence $\bigcap_{x \in V(\Gamma)} \langle g_x \rangle = \langle {g_v}^{h_v} \rangle$ where

$$h_v = \mathrm{lcm}\{\omega_T^{(1)}(v, x) \mid x \in V(\Gamma)\} = \omega_T^{tot}(v),$$

the *total weight* of v in T.

The total weight of v in T is the smallest positive power of g_v belonging to every vertex subgroup.

There is a more economic expression for the total weight. Let $y_1, y_2, \ldots, y_k$ be the vertices of degree 1 in T. Then

$$\omega_T^{tot}(v) = \mathrm{lcm}\{\omega_T^{(1)}(v, y_i) \mid i = 1, 2, \ldots, m\}.$$

How to compute the centre of a GBS-group

Lemma 5. *Let* (Γ, ω) *be a non-elementary GBS-graph with maximal subtree* T*. Assume that no leaf in* T *has distal weight* ± 1*. Then*

$$Z(G) = \bigcap_{e \in E(\Gamma) \setminus E(T)} C_J(t_e),$$

where $J = \bigcap_{x \in V(\Gamma)} \langle g_x \rangle$*. If* $\Gamma = T$*, then* $Z(G) = J$*.*

The centralizers in this formula can be found using:

Lemma 6. *Let* $e = \langle x, y \rangle \in E(\Gamma) \setminus E(T)$ *be* T*-dependent and let* $\omega(e) = (m, n)$ *and* $\omega_T(x, y) = (a, b)$*. Then*

$$C_{\langle g_x \rangle}(t_e) = \langle g_x^{\operatorname{lcm}(a,m)} \rangle.$$

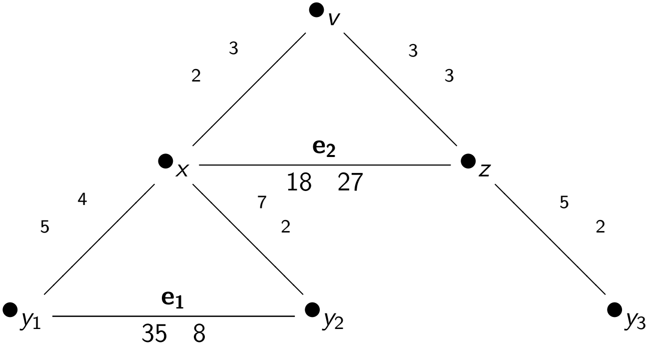
A formula for the centre of a GBS-group

Theorem 4. (A. Delgado, DR, M. Timm.) *Let* (Γ, ω) *be a non-elementary, tree dependent GBS-graph with a maximal subtree* T*. Assume no leaf in* T *has distal weight* ± 1*. Let* v *be any fixed vertex and let the non-tree edges be* $e_i = \langle x_i, y_i \rangle$, $i = 1, 2, \ldots, k$*. Put* $\omega(e_i) = (m_i, n_i)$, $\omega_T(x_i, y_i) = (a_i, b_i)$, $\omega_T(v, x_i) = (c_i, d_i)$, *and* $\ell_i = \mathrm{lcm}(a_i, m_i)$*. Then* $Z(G) = \langle {g_v}^{f_v} \rangle$ *where*

$$f_v = \mathrm{lcm}\{\frac{c_i \ell_i}{\gcd(\ell_i, d_i)},\ \omega_T^{tot}(v) \mid i = 1, 2, \ldots, k\}.$$

Call $f_v = \omega_\Gamma^{tot}(v)$, the *total weight* of v in (Γ, ω).

An example



The two non-tree edges are e_1, e_2 and v is the root of the maximal subtree T, while y_1, y_2, y_3 are the vertices of degree 1 in T. The edges e_1 and e_2 are T-dependent, so (Γ, ω) is tree dependent and $Z(G) \neq 1$.

An example

Read off the required data from the GBS-graph.

$$\omega_T^{tot}(v) = \mathrm{lcm}(\omega_T^{(1)}(v, y_1), \omega_T^{(1)}(v, y_2),\ \omega_T^{(1)}(v, y_3)) = 210.$$

Next

$(m_1, n_1) = \omega(e_1) = (35, 8),\ (m_2, n_2) = \omega(e_2) = (18, 27),$ $(a_1, b_1) = \omega_T(y_1, y_2) = (35, 8),\ (a_2, b_2) = \omega_T(x, z) =$ $(2, 3),\ (c_1, d_1) = \omega_T(v, y_1) = (6, 5),\ (c_2, d_2) =$ $\omega_T(v, x) = (3, 2).$

Hence $\ell_1 = 35, \ell_2 = 18$ and $\omega_\Gamma^{tot}(v) = 1890$. Therefore

$$Z(G) = \langle g_v^{1890} \rangle.$$

Cyclic normal subgroups in GBS-groups

In skew tree dependent GBS-graphs the role of the centre is played by the unique maximum normal cyclic subgroup.

Lemma 7. *Let* (Γ, ω) *be a non-elementary GBS-graph. Then* $G = \pi_1(\Gamma, \omega)$ *has a unique maximal cyclic normal subgroup* $C(G)$.

Proof. Suppose $\{C_i | i \in I\}$ is an infinite ascending chain of cyclic normal subgroups of G. Each C_i is commensurable and hence lies in a vertex subgroup. Hence infinitely many of the C_i lie in some $\langle g_v \rangle$, a contradiction. Hence G has a maximal cyclic normal subgroup C.

It is straightforward to show C is unique.

Corollary. *$C(G) \leq \bigcap_{v \in V(\Gamma)} \langle g_v \rangle = J$ and hence*

$$C(G) = \bigcap_{e \in E(\Gamma) \setminus E(T)} J_{\langle t_e \rangle},$$

where $J_{\langle t_e \rangle}$ is the $\langle t_e \rangle$-core of J.

The subgroup $C(G)$ in a GBS-group can be trivial.

Lemma 8. *Let* $G = \pi_1(\Gamma, \omega)$ *be a non-elementary GBS-group. Then:*

(i) $C(G) \neq 1$ *if and only if* $\pi_1(\Gamma, \omega)$ *is unimodular, i.e.,* (Γ, ω) *is either tree dependent or skew-tree dependent.*

(ii) $1 = Z(G) < C(G)$ *if and only if* (Γ, ω) *is skew tree dependent.*

Computing $C(G)$

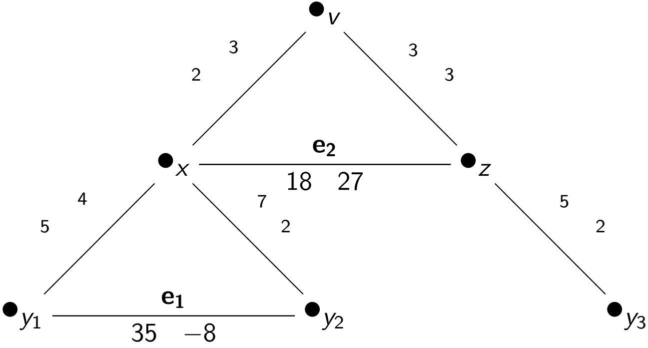
The algorithm to compute the centre of a tree dependent GBS-graph can be applied to a skew tree dependent GBS-graph (Γ, ω), with cores playing the role of centralizers. It will then compute $C(\pi_1(\Gamma, \omega))$.

Theorem 5. (A.Delgado, DR, M.Timm.) *Let* (Γ, ω) *be a non-elementary, skew tree dependent GBS-graph with a maximal subtree* T *having no distal weights* ± 1. *Then if* $G = \pi_1(\Gamma, \omega)$ *and* v *is any vertex* v,

$$C(G) = \langle g_v^{\omega_\Gamma^{tot}(v)} \rangle.$$

An example

Change the weight of edge e_1 in the last example from $(35, 8)$ to $(35, -8)$.



$$Z(G) = 1, \quad C(G) = \langle g_v{}^{\omega_\Gamma^{tot}(v)} \rangle = \langle g_v^{1890} \rangle.$$

GBS-groups and 3-manifold groups

What is the relation between GBS-groups and 3-manifold groups, i.e., the fundamental groups of compact 3-manifolds?

Some examples (W. Heil).

1. $K(m, n) = \langle x, y \mid x^m = y^n \rangle$ is a 3-manifold group.

$\bullet_x \overset{m \qquad\quad n}{\overline{\qquad\qquad\qquad}} \bullet_y$

2. The group $\langle x_1, x_2, x_3 \mid {x_1}^m = x_2^n,\ {x_2}^m = x_3^n \rangle$ is a 3-manifold group iff $|m| = 1$ or $|n| = 1$ or $|m| = |n|$.

$\bullet_{x_1} \overset{m \qquad\quad n}{\overline{\qquad\qquad\qquad}} \bullet_{x_2} \overset{m \qquad\quad n}{\overline{\qquad\qquad\qquad}} \bullet_{x_3}$

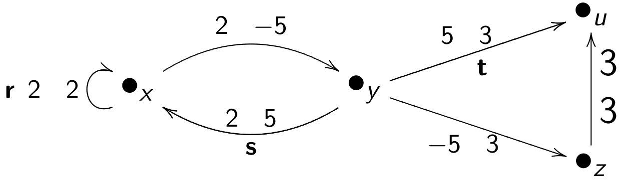
3. $B(m, n) = \langle t, x \mid x^n = (x^m)^t$ is a 3-manifold group iff $|m| = |n|$.

Problem. Find necessary and sufficient conditions on a GBS-graph (Γ, ω) for $\pi_1(\Gamma, \omega)$ to be the fundamental group of a compact 3-manifold.

A GBS-graph (Γ, ω) is called *locally weight constant* if at every vertex v all weights equal c_v and *locally* $\pm$ *weight constant* if all weights at v equal $\pm c_v$ for some constant c_v.

Remarks
Let (Γ, ω) be a GBS-graph. If (Γ, ω) is locally weight constant GBS-graph, it is tree dependent. If it is locally $\pm$ weight constant, it is tree or skew tree dependent, i.e., it is unimodular.

Example The GBS-graph shown is locally $\pm$ weight constant, but not locally weight constant.



The GBS-groups which are 3-manifold groups

Theorem 6. (A. Delgado, DR, M.Timm.) *Let* (Γ, ω) *be a non-elementary GBS-graph. Then the following are equivalent.*

(i) $\pi_1(\Gamma, \omega)$ *is a* 3*-manifold group.*

(ii) $\pi_1(\Gamma, \omega)$ *is an orientable* 3*-manifold group.*

(iii) (Γ, ω) *is locally* $\pm$ *weight constant.*

This explains Heil's examples: $B(m, n)$ is a 3-manifold group if and only if $|m| = |n|$.

3-manifold GBS-group covers

Let (Γ, ω) be a non-elementary GBS-graph. If $\pi_1(\Gamma, \omega)$ is not a 3-manifold group, it may be a quotient of a GBS-group which is a 3-manifold group.

A *3-manifold GBS-group cover* of $\pi_1(\Gamma, \omega)$ is a surjective homomorphism

$$\varphi : \pi_1(\Gamma, \tau) \to \pi_1(\Gamma, \omega)$$

where (Γ, τ) is a GBS-graph such that $\pi_1(\Gamma, \tau)$ is a 3-manifold group, and φ is a *pinch map*, which arises by dividing the weights on certain edges of Γ by common factors.

The GBS-groups with 3-manifold GBS-group covers

Theorem 7. (A. Delgado, DR, M.Timm.) *Let* (Γ, ω) *be a non-elementary GBS-graph. Then the following are equivalent.*

(i) $\pi_1(\Gamma, \omega)$ *has a* 3*-manifold GBS-group cover.*

(ii) $\pi_1(\Gamma, \omega)$ *has an orientable* 3*-manifold GBS-group cover.*

(iii) $\pi_1(\Gamma, \omega)$ *is unimodular, i.e.,* (Γ, ω) *is tree dependent or skew tree dependent.*

Suppose that (Γ, ω) is a non-elementary GBS-graph such that $\pi_1(\Gamma, \omega)$ unimodular. We show how to construct a 3-manifold GBS-group cover of $\pi_1(\Gamma, \omega)$.

Case: (Γ, ω) *is tree dependent.*

Define a new weight function τ on Γ as follows:

$$\tau(e) = (\omega_{\Gamma}^{tot}(e^-), \omega_{\Gamma}^{tot}(e^+), \quad e \in E(\Gamma).$$

Call the GBS-graph (Γ, τ) the *total weight cover* of (Γ, ω).

Constructing 3-manifold GBS-group covers

Clearly the total weight cover is locally weight constant, so $\pi_1(\Gamma, \tau)$ is a compact (orientable) 3-manifold group.

The identity map on Γ and a suitable sequence of pinches yields a surjective homomorphism

$$\varphi : \pi_1(\Gamma, \tau) \to \pi_1(\Gamma, \omega)$$

which is a 3-manifold GBS-group cover of $\pi_1(\Gamma, \omega)$.

The total $\pm$ weight cover

Case: (Γ, ω) *is skew tree dependent*

Let T be a maximal subtree in Γ. We can assume that all weights in T are positive. Write $E(\Gamma)\backslash E(T) = P \cup N$ where P is the set of edges with positive weights and N is the set of remaining edges. Define a new weight function τ on Γ by

$$\tau(e) = (\omega_\Gamma^{tot}(e^-), \omega_\Gamma^{tot}(e^+)), \quad e \in E(T) \cup P,$$

and

$$\tau(e) = (\omega_\Gamma^{tot}(e^-), -\omega_\Gamma^{tot}(e^+)), \quad e \in N.$$

Then (Γ, τ) is a locally $\pm$ weight constant GBS-graph, *the total $\pm$ weight cover* of (Γ, ω). Thus $\pi_1(\Gamma, \tau)$ is a 3-manifold group and we have a 3-manifold GBS-group cover

$$\varphi : \pi_1(\Gamma, \tau) \to \pi_1(\Gamma, \omega)$$

defined by the identity map on Γ and suitable pinches.

Final comments

(i) The 3-manifold GBS-group covers constructed are *minimal* in the sense that all others factor through them.

(ii) The kernels of the covering maps can be computed.