Some designs and binary codes preserved by the simple group Ru of Rudvalis

Bernardo Rodrigues
Joint work with J Moori

School of Mathematics, Statistics and Computer Science
University of KwaZulu-Natal
Durban 4041
South Africa

Groups St. Andrews 2013, University of St. Andrews
8 August 2013

Motivation

- The simple group Ru of Rudvalis is one the 26 sporadic simple groups.
- It has a rank-3 primitive permutation representation of degree 4060 which can be used to construct a strongly regular graph Γ with parameters $v = 4060$, $k = 1755$, $\lambda = 730$ and $\mu = 780$ or its complement a strongly regular $\widetilde{\Gamma} = (4060, 2304, 1328, 1280)$ graph.
- The stabilizer of a vertex u in this representation is a maximal subgroup isomorphic to the Ree group $2_{F_4(2)}$ producing orbits $\{u\}$, Δ_1, Δ_2 of lengths 1, 1755, and 2304 respectively. The regular graphs Γ, $\widetilde{\Gamma}, \Gamma^R, \widetilde{\Gamma}^R$, Γ^S are constructed from the sets Δ_1, Δ_2, $\{u\} \cup \Delta_1$, $\{u\} \cup \Delta_2$, and $\Delta_1 \cup \Delta_2$, respectively.

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Motivation

- If A denotes an adjacency matrix for Γ then $B = J - I - A$, where J is the all-one and I the identity 4060×4060 matrix, will be an adjacency matrix for the graph $\widetilde{\Gamma}$ on the same vertices.
- We examine the neighbourhood designs $\mathcal{D}_{1755}$, $\mathcal{D}_{1756}$, $\mathcal{D}_{2304}$, $\mathcal{D}_{2305}$ and $\mathcal{D}_{4059}$ and corresponding binary codes C_{1755}, C_{1756}, C_{2304}, C_{2305}, and C_{4059} defined by the binary row span of A, $A + I$, B, $B + I$ and $A + B$ respectively.

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Background - t-(v, k, λ) Designs

- An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ with **point set** $\mathcal{P}$ and **block set** $\mathcal{B}$ and incidence $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$ is a $t-(v, k, \lambda)$ design if
  - $|\mathcal{P}| = v$;
  - every block $B \in \mathcal{B}$ is incident with precisely **k** points;
  - every **t** distinct points are together incident with precisely λ blocks. t, v, k and λ are non-negative integers; $|\mathcal{B}| = b$ is the number of blocks;

An incidence matrix for $\mathcal{D}$ is a $b \times v$ matrix $A = (a_{ij})$ of 0's and 1's such that

$$a_{ij} = \begin{cases} 1 & \text{if } (p_j, B_i) \in \mathcal{I} \\ 0 & \text{if } (p_j, B_i) \notin \mathcal{I}\ . \end{cases}$$

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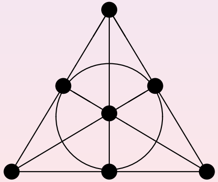
The Fano Plane is a $2-(7,3,1)$ Design

Take $S = \{1,2,3,4,5,6,7\}$ and consider the subsets: $\{1,2,4\}$ $\{2,3,5\}$ $\{3,4,6\}$, $\{4,5,7\}$ $\{5,6,1\}$ $\{6,7,2\}$ $\{7,1,3\}$. We have a $2-(7,7,3,3,1)$-design. We can have a geometrical interpretation of this design as follows:

- The elements of $1,2,3,\ldots,7$ are represented by points and the blocks by lines (6 straight lines and a circle). This is known as the projective plane of order 2.



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Incidence matrix - an example

Blocks (lines)

Points

	b_1	b_2	b_3	b_4	b_5	b_6	b_7
p_1	1	0	0	0	1	0	1
p_2	1	1	0	0	0	1	0
p_3	0	1	1	0	0	0	1
p_4	1	0	1	1	0	0	0
p_5	0	1	0	1	1	0	0
p_6	0	0	1	0	1	1	0
p_7	0	0	0	1	0	1	1

Table : Incidence matrix of the $2-(7,3,1)$ Design

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Background - Graphs

- A graph $\mathcal{G} = (V, E)$, consists of a finite set of vertices V together with a set of edges E, where an edge is a subset of the vertex set of cardinality 2. Our graphs are undirected.
- The valency of a vertex is the number of edges containing the vertex.
- A graph is regular if all the vertices have the same valency; a regular graph is strongly regular of type (n, k, λ, μ) if it has n vertices, valency k, and if any two adjacent vertices are together adjacent to λ vertices, while any two non-adjacent vertices are together adjacent to μ vertices.
- The adjacency matrix $A(\mathcal{G})$ of $\mathcal{G}$ is the $n \times n$ matrix with

$$(i,j) = \begin{cases} 1 & \text{if } x_i \text{ and } x_j \text{ are adjacent,} \\ 0 & \text{otherwise .} \end{cases}$$

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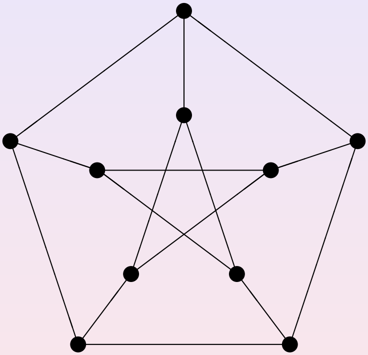
The Petersen Graph is strongly regular



Error-correcting codes

- Let F be any set of size q and let F^n denote the set of n-tuples of elements of F (usually here F will be a finite field). Call the elements of F^n vectors.
- A ***q*-ary code** C of length n is a set of elements of F^n, called codewords or vectors, and written $x_1 x_2 \dots x_n$, or $(x_1, x_2, \dots, x_n)$, where $x_i \in F$ for $i = 1, \dots, n$.

Definition

Let $v = (v_1, v_2, \dots, v_n)$ and $w = (w_1, w_2, \dots, w_n)$ be in F^n. The Hamming distance, $d(v, w)$, between v and w is the number of coordinate places in which they differ:

$$d(v, w) = |\{i | v_i \neq w_i\}|.$$

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Error Correcting Codes

Definition

The minimum distance $d(C)$ *of a code* C *is the smallest of the distances between distinct codewords; i.e.*

$$d(C) = \min\{d(v, w) | v, w \in C, v \neq w\}.$$

Theorem

If $d(C) = d$ *then* C *can detect up to* $d - 1$ *errors or correct up to* $\lfloor (d-1)/2 \rfloor$ *errors.*

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Linear Codes

- A code C over the finite field $F = \mathbf{F}_q$ of order q, of length n is linear if C is a subspace of $V = F^n$. If $\dim(C) = k$ and $d(C) = d$, then we write $[n, k, d]$ or $[n, k, d]_q$ for the q-ary code C.
- If C is a q-ary $[n, k]$ code, a generator matrix for C is a $k \times n$ array obtained from any k linearly independent vectors of C.
- Let C be a q-ary $[n, k]$ code. The dual code of C is denoted by $C^\perp$ and is given by

$$C^\perp = \{v \in F^n | (v, c) = 0 \text{ for all } c \in C\}.$$

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Linear Codes- Continued

- A check matrix for C is a generator matrix H for $C^{\perp}$.
- Two linear codes of the same length and over the same field are isomorphic if they can be obtained from one another by permuting the coordinate positions.
- An automorphism of a code C is an isomorphism from C to C.
- Any code is isomorphic to a code with generator matrix in standard form, i. e. the form $[I_k \mid A]$; a check matrix then is given by $[-A^T \mid I_{n-k}]$. The first k coordinates are the information symbols and the last $n-k$ coordinates are the check symbols.

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A preliminary result

Result

Let G be a finite primitive permutation group acting on the set Ω *of size n. Let* $\alpha \in \Omega$*, and let* $\Delta \neq \{\alpha\}$ *be an orbit of the stabilizer* G_α *of* α*. If* $\mathcal{B} = \{\Delta^g \mid g \in G\}$ *and, given* $\delta \in \Delta$*,* $\mathcal{E} = \{\{\alpha, \delta\}^g \mid g \in G\}$*, then* $\mathcal{D} = (\Omega, \mathcal{B})$ *forms a symmetric* 1-$(n, |\Delta|, |\Delta|)$ *design. Further, if* Δ *is a self-paired orbit of* G_α *then* $\Gamma = (\Omega, \mathcal{E})$ *is a regular connected graph of valency* $|\Delta|$*,* $\mathcal{D}$ *is self-dual, and G acts as an automorphism group on each of these structures, primitive on vertices of the graph, and on points and blocks of the design.*

In fact one can use any union of orbits of a point-stabilizer in this construction, and this is the approach that we will adopt in the paper.

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The Rudvalis group Ru

The primitive representations Ru are listed in Table 2. The first column gives the ordering of the primitive representations; the second gives the maximal subgroups; the third gives the degree (the number of cosets of the point stabilizer);

No.	Max. sub.	Deg.	No.	Max. sub.	Deg.
1	$^{2}F_4(2)$	4060	9	$L_2(29)$	11980800
2	$(2^6{:}U_3 3){:}2$	188500	10	$5^2{:}4S_5$	12160512
3	$(2^2 \times S_z(8)){:}3$	417600	11	$3^{\cdot}A_6{\cdot}2^2$	33779200
4	$2^{3+8}{:}L_3(2)$	424125	12	$5_+{}^{1+2}{:}[2^5]$	36481536
5	$U_3(5){:}2$	579072	13	$L_2(13){:}2$	66816000
6	$2^{\cdot}2^{4+6}{:}S_5$	593775	14	$A_6{}^{\cdot}2^2$	101337600
7	$L_2(25){\cdot}2^2$	4677120	15	$5{:}4 \times A_5$	121605120
8	A_8	7238400			

Table : Maximal subgroups of Ru

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The graphs, designs and codes

- The above Table shows that there is just one class of maximal subgroups of Ru of index 4060. The stabilizer of a vertex u in this representation is a maximal subgroup isomorphic to $2_{F_4(2)}$, producing orbits $\{u\}$, Δ_1, and Δ_2 of lengths 1, 1755 and 2304 respectively.
- The regular graphs $\Gamma, \Gamma^R, \widetilde{\Gamma}, \widetilde{\Gamma}^R$ are constructed from the sets Δ_1, $\{u\} \cup \Delta_1$, Δ_2 and $\{u\} \cup \Delta_2$, respectively.
- The binary codes $C_{1755}, \ C_{1756}, \ C_{2304}, \ C_{2305}$ whose properties we will be examining are obtained as described below.

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The graphs, designs and codes

- The rows of an adjacency matrix A for Γ give the blocks of the neighbourhood design of Γ which we will denote $\mathcal{D}_{1755}$. Notice that $\mathcal{D}_{1755}$ is a self-dual symmetric 1-$(4060, 1755, 1755)$ design. We write C_{1755} to denote the binary code spanned by the rows of $\mathcal{D}_{1755}$.
- From the rows of an adjacency matrix $A + I$ of the reflexive graph Γ^R we obtain the self-dual symmetric 1-$(4060, 1756, 1756)$ design $\mathcal{D}_{1756}$, and the binary code C_{1756}.
- The rows of an adjacency matrix B for $\widetilde{\Gamma}$ yield the neighbourhood 1-$(4060, 2304, 2304)$ design $\mathcal{D}_{2304}$. This is a self-dual symmetric design, and the binary row span of gives the code C_{2304}.

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The graphs, designs and codes

- From the rows of an adjacency matrix $B + I$ of the reflexive graph $\widetilde{\Gamma}^R$ we get the self-dual symmetric 1-$(4060, 2305, 2305)$ design $\mathcal{D}_{2305}$. We write C_{2305} to denote the binary code of $\mathcal{D}_{2305}$.

Results

Lemma

Let G be the Rudvalis group Ru and $\mathcal{D}_i$ and C_i where $i \in \{1755, 2305, 4059\}$ be the designs and binary codes constructed from the primitive rank-3 permutation action of G on the cosets of $2_{F_4(2)}$. Then

(i) $\mathrm{Aut}(\mathcal{D}_{1755}) = \mathrm{Aut}(\mathcal{D}_{2305}) = \mathrm{Ru}$ *and* $\mathcal{D}_{1755}$ *is the unique point-primitive and flag-transitive symmetric design on 4060 points* .

(ii) $C_{1755} = C_{2305} = V_{4060}(\mathbb{F}_2)$.

(iii) $\mathrm{Aut}(C_{1755}) = \mathrm{Aut}(C_{2305}) = S_{4060}$.

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Sketch of the proof

Proof: (i)

- The definition of Ω and $\mathcal{B}$ emerges from Result 1.3, and from this it is clear that $G \subseteq \mathrm{Aut}(\mathcal{D}_{1755})$.
- It follows from Result 1.3, and also from the Atlas [1, p.126] that G acts primitively on both Ω and $\mathcal{B}$ of degree $|\Omega| = |\mathcal{B}| = 4060$, and the stabilizer of a vertex u (point) has exactly three orbits in Ω.
- G_u fixes setwise each of $\{u\}$, Δ_1 and $\Omega \setminus (\Delta_1 \cup \{u\}) = \Delta_2$ and these are all possible G_u-orbits.
- $\mathcal{D}_{1755}$ is a point primitive, symmetric 1-design. It remains to show that $G = \mathrm{Aut}(\mathcal{D}_{1755})$.
- Now $G \subseteq \mathrm{Aut}(\mathcal{D}_{1755}) \subseteq S_{4060}$, so $\mathrm{Aut}(\mathcal{D}_{1755})$ is a primitive permutation group on Ω of degree 4060. Moreover, $\mathrm{Aut}(\mathcal{D}_{1755})_u$ must fix Δ_1 setwise, and hence $\mathrm{Aut}(\mathcal{D}_{1755})_u$ also has orbits of lengths 1, 1755, and 2304 in Ω.

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Sketch of the proof

- The only primitive group of degree 4060, such that $\mathrm{Aut}(\mathcal{D}_{1755})_u$ can have orbit lengths 1, 1755, and 2304 is Ru, see [3, Theorem 18].
- $G = \mathrm{Aut}(\mathcal{D}_{1755})$. Since $\mathcal{D}_{2305} = \tilde{\mathcal{D}}_{1755}$, we deduce that $\mathrm{Aut}(\mathcal{D}_{2305}) = \mathrm{Aut}(\mathcal{D}_{1755}) = \mathrm{Ru}$.
- Recall that there is a unique class of maximal subgroups of Ru of type $2_{F_4(2)}$. Now, given a subgroup K in that class, its normalizer is twice bigger in Ru, meaning that there are exactly two subgroups $2_{F_4(2)}$ that contain K, and so we derive a contradiction.
- Thus, we conclude that there is a unique 1-$(4060, 1755, 1755)$ symmetric design invariant under Ru, and since the block stabilizer acts transitively on the points of the block the claim on flag-transitivity holds.

■

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The code of the graph Γ^R

Lemma

For Ru *of degree 4060, the automorphism group of the graph* Γ^R *or design* $\mathcal{D}_{1756}$ *is a non-abelian finite simple group of order 145926144000. Moreover this group is isomorphic to the simple sporadic group* Ru.

Proof: This follows readily by computations with Magma. ■

Lemma

The group Ru *is the automorphism group of the* $[4060, 29, 1756]_2$ *code* C_{1756} *obtained from* $\mathcal{D}_{1756}$. *The code* C_{1756} *is self-orthogonal doubly-even . Its dual is a* $[4060, 4031, 4]_2$ *code. Moreover,* $\jmath \in C_{1756}$.

The code of the graph $\widetilde{\Gamma}$

Lemma

For Ru of degree 4060, the automorphism group of the design $\mathcal{D}_{2304}$ is isomorphic to the group Ru.

Proof: Since $\mathcal{D}_{2304} = \tilde{\mathcal{D}}_{1756}$, we have $\mathrm{Aut}(\mathcal{D}_{2304}) = \mathrm{Aut}(\tilde{\mathcal{D}}_{1756}) = \mathrm{Aut}(\mathcal{D}_{1756})$. Now the proof follows from Lemma 1.5. ■

Lemma

The group Ru is the automorphism group of C_{2304}. The code C_{2304} is self-orthogonal doubly-even, with minimum weight 1792.Its dual is a $[4060, 4032, 4]_2$. Moreover, Ru acts irreducibly on C_{2304} as an $\mathbb{F}_2$-module, $C_{2304} \subset C_{1756}$, and $\mathrm{Aut}(C_{2304}) = \mathrm{Ru}$.

Sketch of the proof

Proof:

- Use the strong regularity of $\widetilde{\Gamma}$ to show that the code C_{2304} is self-orthogonal.
- Notice first that C_{2304} is obtained from the strongly regular graph $\widetilde{\Gamma}$ with parameters $(4060, 2304, 1328, 1280)$ and intersection matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 2304 & 1328 & 1280 \\ 0 & 975 & 1024 \end{bmatrix}.$$

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Sketch of the proof

- It can be seen from Figure 1 below that if we fix a vertex v in $\widetilde{\Gamma}$ we can divide the remaining vertices into two sets, namely $\widetilde{\Gamma}'$ of size 2304 and $\widetilde{\Gamma}''$ of size 1755, with $\widetilde{\Gamma}'$ being the set of vertices adjacent to v, and $\widetilde{\Gamma}''$ the set of vertices non-adjacent to v.
- Now, from the second column of the above matrix we deduce that each vertex in $\widetilde{\Gamma}'$ is adjacent to v and to 1328 other vertices in $\widetilde{\Gamma}'$, thus to 975 vertices in $\widetilde{\Gamma}''$ while from the third column shows that a vertex in $\widetilde{\Gamma}''$ is adjacent to 1280 vertices in $\widetilde{\Gamma}'$, and so to 1024 vertices in $\widetilde{\Gamma}''$.

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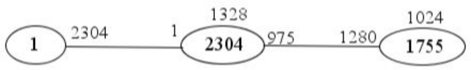
Sketch of the proof

- The structure of the graph and the orbit joins are summarized in the following diagram.

Figure : Number of joins between orbits of a stabilizer



- The valency 2304 ensures that generating codewords have length zero (mod 2) and the 1328 and the 1280 ensure that (*i*) any two generating codewords have an even number of non-zero entries in common, and (*ii*) that any two generating codewords are orthogonal to one another.
- Hence C_{2304} is self-orthogonal, and since all non-zero codewords have weights divisible by 4, it follows that C_{2304} is doubly-even.

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Sketch of the proof

- $$W_{C_{2304}} = 1 + 188500\, x^{1792} + 4677120\, x^{1952} + 38001600\, x^{1984} + 95769600\, x^{2016} + 95597775\, x^{2048} + 33779200\, x^{2080} + 417600\, x^{2240} + 4060\, x^{2304}.$$
- Moreover, the blocks of $\mathcal{D}_{2304}$ are of even size, so $\jmath$ meets evenly every vector of C_{2304}, so $\jmath \in {C_{2304}}^{\perp}$. It can be deduced from [2, Section 3] that the 2-rank of $\widetilde{\Gamma}$ is 28, and so the dimension of C_{2304} follows.
- If $\alpha \in \mathrm{Aut}(C_{2304})$, then since $\alpha(\jmath) = \jmath$ and $C_{1756} = \langle C_{2304}, \jmath \rangle$, we have $\alpha \in \mathrm{Aut}(C_{1756})$. So that $\mathrm{Aut}(C_{2304}) \subseteq \mathrm{Aut}(C_{1756})$.
- Arguing similarly as Lemma 1.6 we show that $\mathrm{Aut}(C_{2304}) = \mathrm{Ru}$. ■

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