The Asphericity of Injective Labeled Oriented Trees

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Introduction

Joint work with Jens Harlander (Boise, Idaho, USA)

The Whitehead-Conjecture

Whitehead-Conjecture [1941]:

(WH): Let L be an aspherical 2-complex.
Then $K \subset L$ is also aspherical.

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Labeled Oriented Trees

A LOG (labeled oriented graph) is a finite oriented graph, where the edges are labeled with vertex labels.

For example

A LOG gives a finite presentation:
Vertices $\longleftrightarrow$ Generators, Edges $\longleftrightarrow$ Relators
A *LOG-presentation*. (There is also a *LOG-complex*)

In our example:
$\langle a, b, c, d, e \mid ac = cb, bd = dc, db = bc, da = ae \rangle$

A LOT (labeled oriented tree) is a LOG which is a tree.

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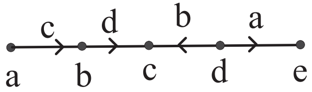

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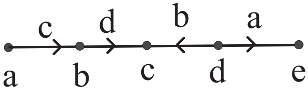

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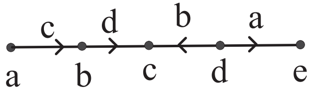

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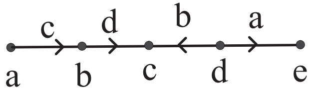

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Theorem (Howie 1983): Let L be a finite 2-complex and $e \subset L$ a 2-cell.

If $L \overset{3}{\nearrow\!\searrow} * \quad \Rightarrow \quad L - e \overset{3}{\nearrow\!\searrow} K$ and K is a LOT complex.

Andrews-Curtis Conjecture (AC): Let L be a finite, contractible 2-complex. Then $L \overset{3}{\nearrow\!\searrow} *$.

Corollary: (AC), LOT complexes are aspherical $\Rightarrow$ There is no finite counterexample $K \subset L$, L contractible, to (WH).
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A nonaspherical LOT complex is a counterexample to (WH):

Any LOT complex is a subcomplex of an aspherical 2-complex (add $x_1 = 1$ as a relator. Can then be 3-deformed to a point).

Hence: The asphericity of LOTs is interesting for (WH)!

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Spherical diagrams

$f\colon C \to K^2$ is a *spherical diagram*, if C is a cell decomposition of the 2-sphere and open cells are mapped homeomorphically.

If K is non-aspherical then there exists a spherical diagram which realizes a nontrivial element of $\pi_2(K)$.

A spherical diagram $f\colon C \to K^2$ is *reducible*, if there is a pair of 2-cells in C with a common edge t, such that both 2-cells are mapped to K by folding over t.

A 2-complex K is said to be *diagrammatically reducible* (DR), if each spherical diagram over K is reducible.

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A spherical diagram $f\colon C \to K^2$ is *vertex reducible*, if there is a pair of 2-cells in C with a common vertex P, such that both 2-cells are mapped to K by folding over P.

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A LOT is called *compressed* if every relator contains 3 different generators.

A LOT is called *boundary-reducible* if there is a boundary vertex which does not appear as edge label.

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A result

Let P be a LOT. A *Sub-LOT* Q of P is a subtree of P with at least one edge such that it is a LOT itself (each edge label of Q is also a vertex label of Q).

Theorem (Huck/Rosebrock 2001): If a compressed injective LOT P does not contain a boundary-reducible Sub-LOT then the LOT-complex $K(P)$ is DR.

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Idea of Proof

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The *Whitehead-Graph* $W(P)$ is the boundary of a regular neighborhood of the only vertex of $K(P)$.

Consists of a pair of vertices x_i^+ (beginning) and x_i^- (end) for each generator x_i.

The *positive graph* $L \subset W(P)$ is the full subgraph on the vertices $x_1^+, \dots, x_n^+$, the *negative graph* $R \subset W(P)$ is the full subgraph on the vertices $x_1^-, \dots, x_n^-$.

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The *weight test* is satisfied for $K(P)$ if there is a real number assigned to each edge of the Whiteheadgraph $W(P)$ (a *weight*), such that

1. the sum of the weights of every reduced cycle is ≥ 2 and
2. For every 2-cell $D \in K(P)$ whose boundary consists of d edges the sum of the weights of the corners of $W(P)$ that correspond to the corners of D is less than or equal to $d - 2$.

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A *reorientation* of a LOT P is a LOT Q that arises from P by changing the orientation of a subset of the edges of P.

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Lemma 1: If the positive graph and the negative graph of a compressed injective LOT P are trees then any reorientation of P is DR.

Proof: If the positive and the negative graph are trees then the weight test is satisfied which implies DR. A reorientation leads to the same corners in a 2-cell:

The weight test depends on the Whiteheadgraph and on the edges each 2-cell contributes to the Whiteheadgraph only. $\square$

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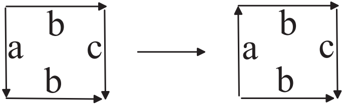

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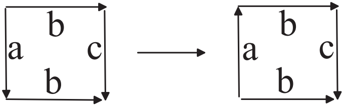

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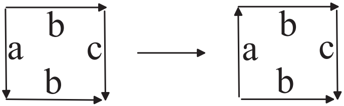

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The condition: "*does not contain a boundary-reducible Sub-LOT*" may be omitted:

Theorem: (Harlander/Rosebrock 2013): Let P be an injective LOT. Then $K(P)$ is aspherical.

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Lemma: STALLINGS (1987) *Each cell decomposition of the 2-sphere contains at least two consistent items.*

Consistent item is a source, a sink or a consistently oriented region.

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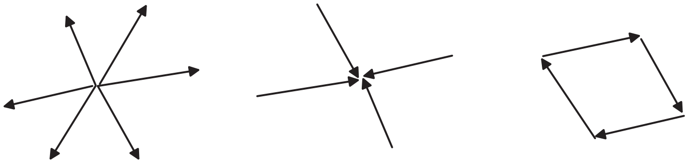

P/T

Let $P = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_m \rangle$ be a LOT-presentation and $T = \{T_1, \ldots, T_n\}$ a set of sub-LOT presentations.

Define

$$P/T = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_m, U_1, \ldots, U_n \rangle$$

where U_i is the set of words of exponent sum 0 in the generators and their inverses of T_i.

Words in $U_1 \cup \ldots \cup U_n$ are called T^*-*relations*.

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Admissible Cycles

A cycle $\alpha = \alpha_1 \ldots \alpha_q$ in the Whitehead graph $W(P/T)$, each α_i being a corner of $W(P/T)$, is called *admissible* if

1. At least one corner α_i comes from a relation which is not a T^*-relation,
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The relative Stallings-test

The presentation P/T is said to *satisfy the relative Stallings-test*, if there is no admissible homology reduced cycle in the positive graph or in the negative graph of $W(P/T)$.

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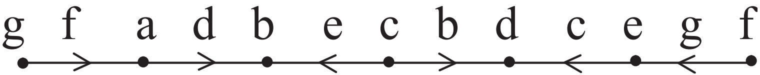

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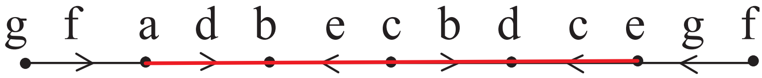

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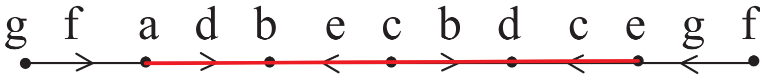

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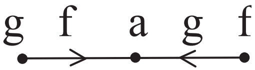
The asphericity of injective LOTs

Let P' be the LOT obtained by collapsing sub-LOTs in P.



Lemma 2 of Huck/Rosebrock above implies that there is a reorientation Q' of P', such that the positive and the negative Whitehead graph of $K(Q')$ are trees.

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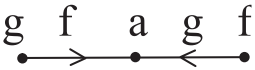

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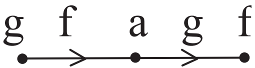

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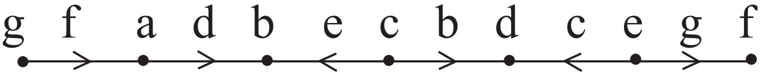
Q is a reorientation of P such that edge orientations coincide with Q' on $Q - T$.



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Lemma: *Q/T satisfies the relative Stallings-test.*

Proof: Relators in Q have exponent sum zero and therefore relators in Q/T also. It remains to show that there are no admissible homology reduced cycles in $W^+(Q/T)$ or $W^-(Q/T)$. This follows from $W^+(Q')$ or $W^-(Q')$ being trees. $\square$

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Let S be the set of edge labels on those edges that change orientation by passing from P to Q. In the example $S = \{g\}$.

Let $(P/T)_S$ be the presentation P/T where each x_i is replaced by x_i^{-1} if $x_i \in S$.

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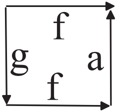

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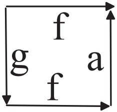

1. The Whitehead graphs $W((P/T)_S)$ and $W(Q/T)$ are equal. Also, the Whitehead graphs $W(P'_S)$ and $W(Q')$ are equal.

2. Let P_S be the presentation P where each x_i is replaced by x_i^{-1} if $x_i \in S$. The 2-complexes $K(P)$ and $K(P_S)$ are homeomorphic.

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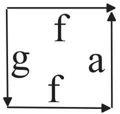

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Lemma: *If $f\colon C \to K((P/T)_S)$ is a vertex reduced spherical diagram then $f(C)$ is contained in $K((T/T)_S)$.*

Idea of proof: Assume $f\colon C \to K((P/T)_S)$ is vertex reduced and $f(C)$ is not contained in $K((T/T)_S)$. Let $E \subset C$ be a maximal region which maps to $P - T$. Glue a disc in each boundary component of E to get a vertex reduced spherical diagram $f'\colon C' \to K((P/T)_S)$ with admissible vertex cycles. C' has no sink and source vertices, but consistently oriented regions may appear.

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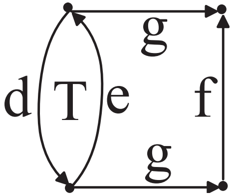
Labeled Oriented Trees



Erase an edge.

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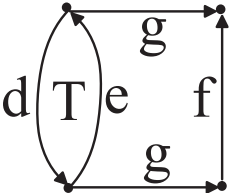

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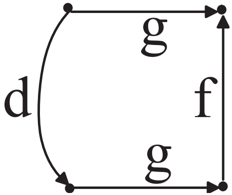

No consistently oriented region, so we have a contradiction to Stallings Lemma. □

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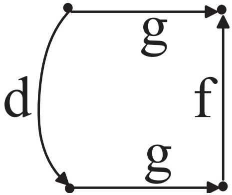

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It now follows that $K(P)$ is aspherical:
Suppose $f\colon C \to K(P)$ is a vertex reduced spherical diagram.

$K(T)$ is aspherical by induction hypothesis so $f(C)$ is not contained in $K(T)$.

$K(P)$ and $K(P_S)$ are homeomorphic, so we have a vertex reduced spherical diagram $f'\colon C' \to K(P_S)$ where $f'(C')$ is not contained in $K(T_S)$.

$K(P_S)$ is a sub-complex of $K((P/T)_S)$, so we have a vertex reduced spherical diagram $f'\colon C' \to K((P/T)_S)$, where $f'(C')$ is not contained in $K((T/T)_S)$.

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  Proceedings of the Edinburgh Math. Soc. 44 (2001).
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