Real character degrees

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Let G be a finite group.

Notation:

$\mathrm{Irr}(G)$ denotes the set of ordinary irreducible complex characters.
$\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$ is the set of irreducible character degrees.

Definition

We say that $\chi \in \mathrm{Irr}(G)$ is a real valued character if $\chi(g) \in \mathbb{R}$ for all $g \in G$.

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$\mathrm{cd}_{\mathbb{R}}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G) \text{ real valued}\}$ is the set of the real valued irreducible character degrees.
If $N \lhd G$, it is clear that $\mathrm{cd}_{\mathbb{R}}(G/N)$ is a subset of $\mathrm{cd}_{\mathbb{R}}(G)$.

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If $N \lhd G$, it is clear that $\mathrm{cd}_{\mathbb{R}}(G/N)$ is a subset of $\mathrm{cd}_{\mathbb{R}}(G)$.

Theorem

If $|\mathrm{cd}(\mathrm{G})| \leq 3$, *then* G *is solvable.*

Theorem (Navarro, Tiep)

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The abelian groups are the only groups with all their irreducible characters linear.

Theorem (Chillag-Mann)

Let G be a finite group. Let $P \in Syl_2(G)$. All real valued irreducible characters of G are linear if and only if

$$G = A \times P$$

where $\mathrm{cd}_{\mathbb{R}}(P) = \{1\}$ and A is the normal 2-complement which is abelian.

We say that a Sylow 2-subgroup P is of the *Chillag-Mann type* if $\mathrm{cd}_{\mathbb{R}}(P) = \{1\}$.

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We recall the Ito-Michler's theorem that shows us when a prime number is missing in $\chi(1)$ for all $\chi \in \mathrm{Irr}(G)$.

Theorem (Ito-Michler)

Let p be a prime number. All nonlinear irreducible characters of G have p'-degree if and only if G has a normal, abelian Sylow p-subgroup.

Theorem (Dolfi, Navarro, Tiep; Isaacs, Navarro)

Let G be a finite group, and let P be a Sylow p-subgroup of G.

1. *All nonlinear real valued irreducible characters of G have odd degree if and only if $P \lhd G$ of the Chillag-Mann type.*
2. *Suppose that $O^{2'}(G) = G$. If all nonlinear real valued irreducible characters of G have p'-degree where p is an odd prime, then $P \lhd G$ and $P' \subseteq Z(G)$.*

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Theorem (Thompson)

Suppose that p divides $\chi(1)$ for every nonlinear $\chi \in \mathrm{Irr}(G)$, where p is a prime. Then G has a normal p-complement.

Theorem (Navarro, S, Tiep; Navarro)

Suppose that G is a finite group.

1. *2 divides $\chi(1)$ for all $\chi \in \mathrm{Irr}(G)$ nonlinear real valued if and only if G has a normal 2-complement.*
2. *Suppose that G is a p-solvable group, where p is an odd prime. Assume that $O^{2'}(G) = G$. If p divides $\chi(1)$ for every nonlinear $\chi \in \mathrm{Irr}(G)$ real valued then G is a p'-group.*

Note that in the p odd case, the condition $O^{2'}(G) = G$ can not be removed. It is sufficient to consider the group $G = PSL(2, 27) \rtimes C_3$ whose order is $|G| = 2^2 \cdot 3^4 \cdot 7 \cdot 13$ and whose set of real character degrees is

$$\mathrm{cd}_{\mathbb{R}}(G) = \{1, 78, 27, 84\}.$$

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Theorem (Navarro, S, Tiep)

Let P be a Sylow 2-subgroup of G. Then every nonlinear real valued irreducible character degree of G is power of 2 if and only of G has a normal 2-complement K and P acts trivially on K'.

Theorem (Isaacs-Passman)

Let G be a finite group. Suppose that every nonlinear irreducible character of G has prime degree. Then

1. *G is solvable.*
2. *$\mathrm{cd}(G) \subseteq \{1, p, q\}$ where p and q are distinct primes and $G'' = 1$.*

Theorem (Dolfi, Pacifici, S)

Let G be a finite group. Suppose that every nonlinear real valued irreducible character of G has prime degree. Then

1. *G is solvable; and*
2. *The set of real character degrees is contained in a set of the kind $\{1, 2, p\}$, where p is and odd prime; and*
3. *$l_2(G) \leq 2$ and $l_{2'}(G) \leq 1$.*

Definition

An element $x \in G$ is *real* if $\chi(x) \in \mathbb{R}$ for all $\chi \in \mathrm{Irr}(G)$ (or equivalently if x and x^{-1} are G-conjugate.)

Notation:

$\mathrm{Re}(G)$ denotes the set of real elements of G.
$\mathrm{cs}_{\mathbb{R}}(G) = \{|x^G| \,|\, x \in \mathrm{Re}(G)\}$ is the set of real conjugacy class sizes.

Theorem (Chillag, Mann)

Let G be a finite group. Let $P \in Syl_2(G)$. All real elements of G are central if and only if

$$G = A \times P$$

where A is the normal 2-complement and $\mathrm{Re}(P) \subseteq Z(P)$.

Theorem (Dolfi, Navarro, Tiep)

Let G be a group. All real classes of G have odd size if and only if G has a normal Sylow 2-subgroup P and $\mathrm{Re}(P) \subseteq Z(P)$.

From the previous theorem, it follows that G is solvable. The p odd version for the previous result is a difficult problem that has required new complex techniques in order to be proved.

Theorem (Guralnick, Navarro, Tiep)

Let G be a finite group, and let p and odd prime. If $p = 3$, assume in addiction that G has no composition factor isomorphic to $\mathrm{SL}_3(2)$. If p does not divide $|x^G|$ for every real class x^G of G, then G is p-solvable.

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Theorem (Dolfi, Pacifici, S)

Assume that all noncentral real conjugacy classes of a group G have prime size. Then

1. *G is solvable.*
2. *There exists a prime p such that $\mathrm{cs}_{\mathbb{R}}(G) \subseteq \{1, 2, p\}$.*
3. *G has either a normal 2-complement or a normal Sylow 2-subgroup.*

Theorem (Guralnick, Navarro, Tiep)

Let G be a group and let p be an odd prime dividing the order of G. If $p = 3$, assume in addiction that G has no composition factor isomorphic to $\mathrm{SL}_3(2)$. If p does not divide $|x^G|$ for all real element $x \in G$ then p does not divide $\chi(1)$ for all $\chi \in \mathrm{Irr}(G)$ real valued.

The converse is not true. It is sufficient to consider the group $G = S_3$ whose set of real degrees is $cd_{\mathbb{R}}(G) = \{1, 2\}$ and whose set of real class sizes is $cs_{\mathbb{R}}(G) = \{1, 2, 3\}$.

Theorem (Casolo, Dolfi)

Let G be a finite group and let p and q be distinct primes. Suppose there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi(1)$ is divisible by pq. Then there exists $g \in G$ such that $|g^G|$ is divisible by pq.

This together with the Ito-Michler's theorem tells us that the prime degree graph is a subgraph of the prime class sizes graph.

Unfortunately the above result is not true for real character degrees and real conjugacy classes.

It suffices to consider the group $G = (C_2 \times C_4 \times A_4) \rtimes C_4$ whose order is 384 and $cs_{\mathbb{R}}(G) = \{1, 3, 8\}$ and $cd_{\mathbb{R}}(G) = \{1, 2, 3, 6\}$.

Thus, the above example show us that the prime graph associated to the real conjugacy classes sizes is not a subgraph of the prime graph associated to the real character degrees.

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