On a maximal subgroup of $O_{10}^{+}(2)$

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Quotation

As for everything else, so for a mathematical theory: beauty can be perceived but not explained - Arthur Cailey.

Abstract

The group $\overline{G} = 2^8{:}O_8^+(2)$ is a group of order 44590694400 and also a maximal subgroup of index 527 of $O_{10}^+(2)$. In turn $2^{10+16}{\cdot}O_{10}^+(2)$ is a maximal subgroup of the monster $M = F_1$. The group $\overline{G}$ has three inertia factor groups namely, $O_8^+(2),\ SP(6,2),\ 2^6{:}A_8$ and each is of index $1,\ 120,$ and 135 respectively in $O_8^+(2)$. The aim of this paper is to compute the Fischer Clifford matrices of $\overline{G}$, which together with its partial character tables are used to compute the full character table of $\overline{G}$. There are 53 Clifford matrices with sizes between 1 and 6. We also give the abstract specification of $2^8{:}O_8^+(2)$

- If N and G are groups, then a split extension of N by G is a group $\bar{G}$ that satisfies
  - $N \trianglelefteq \bar{G}$
  - $\bar{G}/N \cong G$
  - $G \leq \bar{G}$ and $N \cap G = \{1_G\}$
- A homomorphism $\rho : G \rightarrow GL(n, \mathbb{F})$ is a representation of degree n of G over $\mathbb{F}$.
- The function $\chi : G \rightarrow \mathbb{F}$ given by $\chi(g) = trace(\rho(g))$ is called a character of G over $\mathbb{F}$
- If $N \trianglelefteq \bar{G}$ and $\theta \in Irr(N)$, then $I_{\bar{G}}(\theta) = \{\bar{g} \in \bar{G} : \theta^{\bar{g}} = \theta\}$ is an inertia group of θ in $\bar{G}$
- If we have inertia group $\bar{H} = I_{\bar{G}}(\theta)$ then the corresponding inertia factor $H = \bar{H}/N$.

Symmetric Bilinear Form

A symmetric bilinear form on a vector space V over $\mathbb{F}_q$ is a function $f(x, y)$ defined for all $x, y \in V$ and taking values in $\mathbb{F}_q$ which first satisfy *linearity* in x that is

$$f(\lambda_1 x_1 + \lambda_2 x_2, y) = \lambda_1 f(x_1, y) + \lambda_2 f(x_2, y),$$

and secondly symmetry that is

$$f(x, y) = f(y, x).$$

If symmetry is satisfied $f(x, y)$ then also becomes linear in y hence we say f is in symmetric bilinear form.

Quadratic Form

- A quadratic form on V is a function $f(x,x)$ defined for $x \in V$ and taking values in $\mathbb{F}_q$ for which we have

$$f(\lambda x + \mu y, \lambda x + \mu y) = \lambda^2 f(x,x) + \lambda\mu f(x,y) + \mu^2 f(y,y)$$

  for some symmetric bilinear form $f(x,y)$.
- The kernel of f is the subspace of all x such that $f(x,y) = 0 \;\; \forall \; y$.
- The nullity and rank of f is the dimension and codimension of its kernel. We say f is non-singular if the nullity of f is zero.
- A subspace of V is said to be totally isotropic for f if $f(x,y) = 0 \; \forall \; x, y$.
- The Witt index of a quadratic form f is the greatest dimension of any totally isotropic subspace for f

Orthogonal Groups

- The general orthogonal group $GO(V, f)$, is the group of linear maps g satisfying $f(u^g, v^g) = f(u, v) \;\; \forall \; u, v \in V$.
- This is written as $GO(n, q)$ or $GO_n(q)$.
- $GO(n, q) \leq PSL(n, q) = L(n, q)$ that fixes the non-singular quadratic form f.
- If $n = 2m$, then $GO(n, q) = GO^{\epsilon}(n, q)$, $\epsilon = \; + \;$ or $-$
- If $\epsilon = +$, then Witt index $= m$

Transvection

- A reflection is defined by the formula :

$$r_v : x \to x - 2\frac{f(x,v)}{f(v,v)}v, \quad v \in V, \quad f(v,v) \neq 0.$$

- If $\frac{1}{2}f(v,v) = Q(v)$, over a group of characteristic 2, $\;||v|| = 1$, an orthogonal transvection is defined by

$$t_v : w \to w + f(w,v)v.$$

- This is a linear map and preserves the quadratic form since

$$\begin{aligned} Q(w + f(w,v)v) &= \frac{1}{2}f(w + f(w,v)v, w + f(w,v)v) \\ &= \frac{1}{2}f(w,w) + f(w, f(w,v)v) + \frac{1}{2}f(f(w,v)v, f(w,v)v) \\ &= Q(w) + f(w,v)^2 + f(w,v)^2Q(v) \\ &= Q(w). \end{aligned}$$

Size of $O_{2m}^{+}(q)$

- $$\begin{aligned} |GO_{2m}^{+}(q)| &= \Pi_{i=1}^{m}(q^{i-1})(q^{i-1}+1)q^{2i-2} \\ &= 2q^{m(m-1)}(q^{m}-1)\Pi_{i=1}^{m-1}(q^{2i}-1). \end{aligned}$$

- $$\begin{aligned} |O_{10}^{+}(2)| &= 2^{20}(2^{5}-1)\Pi_{i=1}^{4}(2^{2i}-1) \\ &= 23499295948800 \end{aligned}$$

From $\mathbb{ATLAS}$ of group representations V3 we get.

- Two 10×10 matrices a, b $\; o(a) = 2, \; o(b) = 20$ and $o(ab) = 21$
- $O_{10}^+(2) = Group(a, b)$
- $2^8{:}O_8^+(2) = Group(a, c)$
- $o(c) = 30, \; o(ac) = 9$
- $N = 2^8 =< 2A^4, 2B^4 >$
- $2^8{:}O_8^+(2) = N_{O_{10}^+(2)}(2A_{135}2B_{120})$

Fischer-Clifford Theory

Let $\overline{G} = N{\cdot}G$, where $N \lhd \overline{G}$ and $\overline{G}/N \cong G$, be a group extension. The character table of $\overline{G}$ can be constructed once we have

- the character tables of the inertia factor groups,
- the fusions of classes of the inertia factors into classes of G,
- the Fischer-Clifford matrices of $\overline{G} = N{\cdot}G$.

Generators of $O_8^+(2)$

- Using GAP we get $GO^+(8,2)$.
- $O_8^+(2) \trianglelefteq GO^+(8,2)$
- 8×8 matrices a, b with $o(a) = 15,\ o(b) = 4$, and $o(ab) = 7$
- $G = Group(a, b)$
- $\det(a) = \det(b) = 1$
- $GO_8^+(2) = SO_8^+(2)$

Table: Conjugacy Classes of $O_8^+(2)$

$[g]_G$	M	$\lvert[g]_G\rvert$	$[g]_G$	M
$1a$	$\begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}$	1	2a	$\begin{pmatrix} 1&0&0&1&1&1&1&1\\ 0&1&1&1&1&0&1&1\\ 0&0&1&0&0&0&1&0\\ 0&0&0&1&1&0&1&0\\ 1&1&0&1&1&1&0&1\\ 0&1&1&1&0&0&1&1\\ 0&1&0&0&1&0&0&0\\ 1&1&1&1&1&0&0&1 \end{pmatrix}$
2b	$\begin{pmatrix} 0&1&0&1&0&1&1&0\\ 0&0&1&0&0&1&0&0\\ 0&0&0&0&0&1&1&0\\ 0&0&1&0&0&1&0&1\\ 0&1&0&1&1&0&0&1\\ 0&0&0&0&0&1&0&0\\ 0&1&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0 \end{pmatrix}$	1575	2c	$\begin{pmatrix} 1&0&1&0&0&1&1&0\\ 0&0&0&1&0&1&1&0\\ 0&1&0&1&1&1&1&1\\ 1&0&1&1&1&1&1&0\\ 0&1&0&1&1&0&0&1\\ 0&0&1&1&1&1&1&0\\ 1&0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0&1 \end{pmatrix}$
2d	$\begin{pmatrix} 0&0&1&0&1&0&0&0\\ 0&1&0&1&1&0&0&0\\ 0&1&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0\\ 1&0&1&0&1&0&1&0\\ 0&1&1&1&1&1&0&0\\ 0&0&1&1&1&1&0&0\\ 1&0&1&0&0&0&1&0 \end{pmatrix}$	3780	2e	$\begin{pmatrix} 0&0&1&0&0&0&0&0\\ 1&1&0&1&1&1&0&0\\ 1&1&0&0&1&1&0&1\\ 1&0&1&0&0&0&0&0\\ 1&0&0&0&0&1&0&0\\ 1&0&0&0&1&0&0&0\\ 1&0&1&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}$

$O_8^+(2)$ has 53 conjugacy classes.

Action of $O_8^+(2)$ on 2^8

- When $G = O_8^+(2)$ acts on $N = 2^8$ we get three orbits of length

$$1,\ 120,\ \ 135$$

- Corresponding point stabilizers

$$O_8^+(2),\ Sp_6(2),\ 2^6{:}A_8$$

- $\chi(O_8^+(2)|2^8) = 3 \times 1a + 35a + 50a + 2 \times 84a$

Table:

$[g]_{O_8^+(2)}$	1a	2a	2b	2c	2d
$\chi(Sp_6(2)\|2^8)$	120	32	24	8	0
$\chi(2^6{:}A_8\|2^8)$	135	31	39	7	15
k	256	64	64	16	16

Conjugacy Calsses of $2^8{:}O_8^+(2)$

- Programme A and Programme B

Table: Conjugacy Classes of $2^4{:}S_6$

$[g]_G$	k	f_j	d_j	w	$[x]_{\overline{G}}$	$\lvert C_{\overline{G}}(x)\rvert$
$1A$	2^8	1	$(0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0)$	$1A$	44 590 694 4
		120	$(0,0,0,0,0,0,1,0)$	$(1,0,1,0,1,0,1,0)$	$2A$	371 589 12
		135	$(0,0,0,0,0,0,0,1)$	$(1,0,0,1,0,0,0,1)$	$2B$	330 301 44
$2A$	2^6	1	$(0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0)$	$2C$	7 077 888
		6	$(0,0,0,0,0,0,1,0)$	$(0,0,0,0,0,0,0,0)$	2D	1 179 648
		9	$(0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0)$	2E	786 432
		48	$(0,0,0,1,0,0,0,0)$	$(1,0,0,0,0,1,1,1)$	4A	147 456
$2B$	2^6	1	$(0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0)$	$2F$	2 949 120
		6	$(0,0,0,0,0,0,1,0)$	$(0,1,0,1,0,0,1,0)$	4B	491 520
		10	$(0,0,0,0,0,0,0,1)$	$(0,1,0,0,0,0,0,0)$	4C	294 912
		15	$(0,0,0,1,0,0,0,0)$	$(0,0,0,0,0,0,0,0)$	2G	196 608
		32	$(0,0,1,0,0,0,0,0)$	$(1,0,0,0,0,1,1,1)$	4D	92 160

- Brauer's Theorem
- Inertia factor groups are

$$O_8^+(2),\ Sp_6(2),\ 2^6{:}A_8$$

Table: The fusion of $2^6{:}A_8$ into $O_8{}^+(2)$

$[x]_{2^6:A_8}$	$\longrightarrow$	$[g_1]_{O_8^+(2)}$	$[x]_{2^6:A_8}$	$\longrightarrow$	$[g_1]_{O_8^+(2)}$
$1A$		$1A$	$2A$		$2B$
$2B$		$2A$	$2C$		$2A$
$2D$		$2C$	$2G$		$2B$

Table: The fusion of $SP(6,2)$ into $O_8^+(2)$

$[x]_{SP(6,2)}$	$\longrightarrow$	$[g_1]_{O_8^+(2)}$	$[x]_{SP(6,2)}$	$\longrightarrow$	$[g_1]_{O_8^+(2)}$
$1A$		$1A$	$2A$		$2B$
$2B$		$2A$	$2C$		$2B$
$2D$		$2E$	$3A$		$3A$
$3B$		$3D$	$3C$		$3E$

Fischer-Clifford Matrices

Table: Fischer-Clifford Matrices

M(g)	M(g)
$M(1A) = \begin{bmatrix} 1 & 1 & 1 \\ 120 & 8 & -8 \\ 135 & -9 & 7 \end{bmatrix}$	$M(2C) = \begin{bmatrix} 1 & 1 \\ 15 & -1 \end{bmatrix}$
$M(2A) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 24 & 8 & -8 & 0 \\ 3 & 3 & 3 & -1 \\ 36 & -12 & 4 & 0 \end{bmatrix}$	$M(2D) = \begin{bmatrix} 1 & 1 \\ 15 & -1 \end{bmatrix}$
$M(2B) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & -2 & -2 & 2 & 0 \\ 30 & 10 & -6 & -2 & 0 \\ 1 & 1 & 1 & 1 & -1 \\ 30 & -10 & 6 & -2 & 0 \end{bmatrix}$	$M(3D) = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$
$M(2E) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 8 & 0 & -8 & 0 \\ 1 & -1 & 1 & 1 \\ 6 & 0 & 6 & -2 \end{bmatrix}$	$M(4D) = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$

Character Table

$$C_1(2B)M_1(2B) = \begin{bmatrix} 1 \\ 4 \\ 11 \\ -5 \\ -5 \\ 10 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 4 & 4 & 4 & 4 \\ 11 & 11 & 11 & 11 & 11 \\ -5 & -5 & -5 & -5 & -5 \\ -5 & -5 & -5 & -5 & -5 \\ 10 & 10 & 10 & 10 & 10 \end{bmatrix}$$

$$C_2(2B)M_2(2B) = \begin{bmatrix} 1 & 1 \\ -5 & 3 \\ -5 & 3 \\ 9 & 1 \\ -11 & 5 \\ 15 & 7 \end{bmatrix} \begin{bmatrix} 2 & -2 & -2 & 2 & 0 \\ 30 & 10 & -6 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 32 & 8 & -8 & 0 & 0 \\ 80 & 40 & -8 & -16 & 0 \\ 80 & 40 & -8 & -16 & 0 \\ 48 & -8 & -24 & 16 & 0 \\ 128 & 72 & -8 & -32 & 0 \\ 240 & 40 & -72 & 16 & 0 \end{bmatrix}$$

$$C_3(2B)M_3(2B) = \begin{bmatrix} 1 & 1 \\ 7 & 3 \\ 14 & 2 \\ 20 & 4 \\ 21 & 1 \\ 21 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & -1 \\ 30 & -10 & 6 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 31 & -9 & 7 & -1 & -1 \\ 97 & -23 & 25 & 1 & -7 \\ 74 & -6 & 26 & 10 & -14 \\ 140 & -20 & 44 & 12 & -20 \\ 51 & 11 & 27 & 19 & -21 \\ 51 & 11 & 27 & 19 & -21 \end{bmatrix}$$

Fusion of $2^8{:}O_8^+(2)$ into $O_{10}^+(2)$

Table: $2^8{:}O_8^+(2) \longrightarrow O_{10}^+(2)$

$[x]_{2^8:O_8^+(2)}$	$\longrightarrow$	$[g_1]_{O_{10}^+(2)}$	$[x]_{2^8:O_8^+(2)}$	$\longrightarrow$	$[g_1]_{O_{1]}^+(2)}$
$1A$		$1A$	$2A$		$2B$
$2B$		$2A$	$2C$		$2A$
$2D$		$2D$	$2E$		$2C$
$2F$		$2B$	$2G$		$2D$
$2H$		$2C$	$2I$		$2C$
$2J$		$2D$	$3A$		$3A$
$3B$		$3C$	$3C$		$3C$
$3D$		$3D$	$3E$		$3B$
$4A$		$4B$	$4B$		$4A$
$4C$		$4B$	$4D$		$4C$
$4E$		$4D$	$4F$		$4D$
$4G$		$4F$	$4H$		$4D$
$4I$		$4E$	$4J$		$4A$
$4K$		$4G$	$4L$		$4E$

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Asante Sana !!!!!!!!!!!!