Counting cyclic identities in specific finite groups

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Let G be a finite group.

Let Y be an n-cycle.

Let $e_1, e_2, \cdots e_n$ be n consecutive edges of Y.

$l_n(G)$: The number of functions $f : Y \to G$, $f(e_1) \cdot f(e_2) \cdots f(e_n) = f(e_n) \cdots f(e_1) = 1$, where 1 is the identity element in G.

$g_i := f(e_i)$.

$$g_1 \cdot g_2 \cdots g_n = g_n \cdot g_{n-1} \cdots g_1 = 1$$

equivalent to:

$$g_1 \cdot g_2 \cdots g_{n-1} = g_{n-1} \cdot g_{n-2} \cdots g_1,$$

$$g_n = g_1^{-1} \cdot g_2^{-1} \cdots g_{n-1}^{-1} =$$

$$= g_{n-1}^{-1} \cdot g_{n-2}^{-1} \cdots g_1^{-1}.$$

g_n is uniquely determined by the elements g_i, where $1 \leq i \leq n - 1$.

Calculation of $l_3(G)$

$l_3(G) = \#\{g_1, g_2 \in G | [g_1, g_2] = 1\}$.

$l_3(G) = \sum_{g \in G} C_g(G) = \sum_{g \in G} \frac{|G|}{[g]_G}$.

By Burnside's Lemma:

$l_3(G) = |G| \cdot |Class(G)|$,

where $|Class(G)|$ is the number of conjugacy classes of the group G.

Calculation of $l_4(G)$

$g_1 \cdot g_2 \cdot g_3 = g_3 \cdot g_2 \cdot g_1$

equivalent to:

$g_3^{-1} \cdot g_1 \cdot g_2 \cdot g_3 = g_2 \cdot g_1 = g_2 \cdot g_1 \cdot g_2 \cdot g_2^{-1}$

equivalent to:

$[g_1 \cdot g_2, g_3 \cdot g_2] = 1$

Therefore:

$l_4(G) = \#\{g_1, g_2, g_3 \in G | [g_1 \cdot g_2, g_3 \cdot g_2] = 1\}$.

For every choice of $g_2 \in G$, there are $l_3(G)$ possible choices for g_1 and g_3.

$l_4(G) = |G| \cdot l_3(G) = |G|^2 \cdot |Class(G)|$.

Nilpotent groups

G is a nilpotent group of class 2 iff there exists the following central series:

$G \triangleright [G, G] \triangleright [G, [G, G]] = \{1\}$.

equivalent to:

$[G, G] \subseteq Z(G)$.

Nilpotent groups of Class 2 with cyclic commutator of order p

Every $g \in G$ has a unique presentation of a form:

$g = a_1^{i_1} \cdot a_2^{i_2} \cdots a_k^{i_k} \cdot x^j \cdot y$, where:

- $0 \leq i_n < p$, for $1 \leq n \leq k$.
- $x \in [G, G]$, and $0 \leq j < p$.
- $x, y \in Z(G)$.
- $a_m \cdot a_l = a_l \cdot a_m \cdot x^{t_{l,m}}$.

Using the form $g_r = a_1^{i_{1,r}} \cdot a_2^{i_{2,r}} \cdots a_k^{i_{k,r}} \cdot x^{j,r} \cdot y_r$:

$g_1 \cdot g_2 \cdots g_{n-1} = g_{n-1} \cdot g_{n-2} \cdots g_1$

implies the following equation:

$$\sum_{1 \le l < m \le k} t_{l,m} \cdot v(l,m) = 0$$

where:

$$v(l,m) = \sum_{r=1}^{n-1} i_{l,r} \cdot \left(\sum_{u=1}^{r-1} i_{m,u} - \sum_{v=r+1}^{n-1} i_{m,v} \right)$$

For odd n:

$$l_n(G) \leq |Z(G)| \cdot p^{\lceil \frac{k}{2} \rceil (n-1)-1} \left(p^{\lfloor \frac{k}{2} \rfloor (n-1)} + p - 1 \right)$$

For even n:

$$l_n(G) \leq |Z(G)| \cdot p^{\lceil \frac{k}{2} \rceil (n-1)} \left(p^{\lfloor \frac{k}{2} \rfloor (n-1)-1} + p - 1 \right)$$

For $k \leq 3$ the inequality turns to equality.

Every non-abelian group G of order p^3 is nilpotent group of class 2 with a cyclic commutator.

$l_n(G) = p^{2n-3}(p^{n-1} + p - 1)$ for every odd n.

$l_n(G) = p^{2n-2}(p^{n-2} + p - 1)$ for every even n.

If G is a nilpotent group of class 2 which order is p^4, then the commutator is cyclic of order p.

$l_n(G) = p^{3n-4}(p^{n-1} + p - 1)$ for every odd n.

$l_n(G) = p^{3n-3}(p^{n-2} + p - 1)$ for every even n.

Metabelian groups

$G = A \rtimes B$, where A and B are abelian groups.

Assume the following condition:

- $\forall g \in G$, such that $g \notin A$, $|C_G(g)| = \frac{|G|}{|[G,G]|}$.

$$l_n(G) = \frac{|A|^{n-1}\left(|B|^{n-1} - 1 + |[G,G]|\right)}{|[G,G]|} \text{ for odd } n.$$

$$l_n(G) = \frac{|A|^{n-1} \cdot |B| \left(|B|^{n-2} - 1 + |[G,G]|\right)}{|[G,G]|} \text{ for even } n.$$

Example:

$G = D_{2k}$, the dihedral group of order $2k$.

If k is odd: $[G, G] = A$, and then:

$$l_n(G) = k^{n-2}\left(2^{n-1} - 1 + k\right) \text{ for odd } n.$$

$$l_n(G) = 2k^{n-2}\left(2^{n-2} - 1 + k\right) \text{ for even } n.$$

If k is even: $[G, G] = A^2$, and then:

$$l_n(G) = k^{n-2}(2^n - 2 + k) \text{ for odd } n.$$

$$l_n(G) = 2k^{n-2}\left(2^{n-1} - 2 + k\right) \text{ for even } n.$$

If $G = A_4$:

$$l_n(G) = 4^{n-2}\left(3^{n-1} + 3\right) \text{ for odd } n.$$

$$l_n(G) = 4^{n-2}\left(3^{n-1} + 9\right) \text{ for even } n.$$

Example of the solvable group $G = S_4$

for odd n:

$$l_n(G) = \frac{4^{n-2} \cdot 6 \cdot \left(3 \cdot 6^{n-2} + 5 \cdot 2^{n-2} - 3\right)}{5}$$

for even n:

$$l_n(G) = 4^{n-2} \cdot 6^2 \cdot \left(2 \cdot 6^{n-4} + 3^{n-4} + 2^{n-3}\right)$$