Primitive permutation groups with finite stabilizers

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Primitive permutation groups

A transitive group G of permutations of a set Ω is primitive if no (proper, non-trivial) equivalence relation on Ω is preserved by G.

An imprimitive permutation group induces a permutation group on the classes of some (proper, non-trivial) equivalence relation.

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Finite primitive permutation groups

Often called the Aschbacher–O'Nan–Scott Theorem, the classification states that every finite primitive permutation group lies in one of the following classes*:

- I affine groups;
- II almost simple groups;

- III(a) simple diagonal action;
- III(b) product action;
- III(c) twisted wreath action.

(* as stated these classes are not mutually exclusive, see for example Liebeck Praeger Saxl: On the O'Nan Scott Theorem for finite primitive permutation groups, 1988)

Finite primitive permutation groups

Often called the Aschbacher–O'Nan–Scott Theorem, the classification states that every finite primitive permutation group G lies in precisely one of the following classes: ($\operatorname{soc} G \cong K^m$ and K simple)

- I affine groups (K is abelian);
- II almost simple groups (K is nonabelian, $m = 1$);

- III Product: K is nonabelian, $m > 1$
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Primitive permutation groups with a finite stabilizer

My theorem states that every infinite primitive permutation group with a finite point-stabilizer lies in precisely one of the following classes:

- II almost simple groups; $K \leq G \leq \mathrm{Aut}\ K$
- III(b) product action; $G \leq H \,\mathrm{Wr}\, S_m$
- IV split extension. $G = K^m.A$

Of course, more information is given. Some highlights:

- K is a finitely generated simple group;
- in III(b), $H \leq \mathrm{Sym}\,(\Gamma)$ is primitive of type II, $m > 1$ is finite and G permutes the components of Γ^m transitively;
- in IV, K^m acts regularly, A is finite and no non-identity element of A induces an inner automorphism of K^m.

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Theorem(SS):

If $G \leq \mathrm{Sym}(\Omega)$ is infinite & primitive with a finite point stabilizer G_α, then G is fin. gen. by elements of finite order & possesses a unique (non-trivial) minimal normal subgroup M; there exists an infinite, non-abelian, fin. gen. simple group K such that $M = K_1 \times \cdots \times K_m$, where $m \in \mathbb{N}$ and each $K_i \cong K$; and G falls into precisely one of:

- IV M acts regularly on Ω, and G is equal to the split extension $M.G_\alpha$ for some $\alpha \in \Omega$, with no non-identity element of G_α inducing an inner automorphism of M;
- II M is simple, and acts non-regularly on Ω, with M of finite index in G and $M \leq G \leq \mathrm{Aut}(M)$;
- III(b) M is non-regular and non-simple. In this case $m > 1$, and G is permutation isomorphic to a subgroup of the wreath product $H \,\mathrm{Wr}_\Delta \,\mathrm{Sym}(\Delta)$ acting in the product action on Γ^m, where $\Delta = \{1, \ldots, m\}$, Γ is some infinite set, $H \leq \mathrm{Sym}(\Gamma)$ is an infinite primitive group with a finite point stabilizer and H is of type (II). Here K is the unique minimal normal subgroup of H.

A classification of primitive permutation groups with finite point stabilizers

Theorem (Aschbacher, O'Nan, Scott, SS)

Every primitive permutation group G with a finite point-stabilizer is permutation isomorphic to precisely one of the following types:

- I *Finite affine*
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  - III(c) *Finite twisted wreath action*
- IV *Denumerable split extension**

**The "twisting homomorphism" needed for the twisted wreath product is from a point stabilizer in S_m to $\mathrm{Aut}(K)$, and its image contains $\mathrm{Inn}(K)$ but in the infinite case K is infinite.*

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Some consequences

Using this classification we obtain a description of those primitive permutation groups with bounded subdegrees. First we need:

Theorem (Schlichting)

Let G be a group and H a subgroup. Then the following conditions are equivalent:

1. *the set of indices* $\{|H : H \cap gHg^{-1}| : g \in G\}$ *has a finite upper bound;*
2. *there exists a normal subgroup* $N \trianglelefteq G$ *such that both* $|H : H \cap N|$ *and* $|N : H \cap N|$ *are finite.*

Bounded subdegrees

Theorem

Every primitive permutation group whose subdegrees are bounded above by a finite cardinal is permutation isomorphic to precisely one of the following types:

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- III *A product:*
  - III(a) *Finite diagonal action*
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Or to put it topologically...

Corollary

Suppose G is totally disconnected and locally compact, and G contains a maximal (proper) subgroup V that is open and compact. If G contains a compact open normal subgroup then the quotient $G/Core_G(V)$ is permutation isomorphic to precisely one of the following types:

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Read: R. G. Möller, “Structure theory of totally disconnected locally compact groups via graphs and permutations”

Countable and subdegree finite

It also gives a classification of the subdegree finite primitive permutation groups that are closed and countable.

Theorem

If G is a closed and countable primitive permutation group, and the subdegrees of G are all finite, then G is permutation isomorphic to precisely one of the following types:

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Regular suborbits

In the 7th Issue of the Kourovka Notebook, A. N. Fomin asked:

Question

What are the primitive permutation groups (finite and infinite) which have a regular suborbit, that is, in which a point stabilizer acts faithfully and regularly on at least one of its orbits.

Theorem

If G is an primitive permutation group which has a regular finite suborbit, then G again falls under the classification.*

(* in fact requiring that G has a finite self-paired suborbit in which a point stabilizer acts regularly but not necessarily faithfully gives you faithfulness for free.)

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Thank you

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"A classification of primitive permutation groups with finite stabilizers"
http://arxiv.org/abs/1109.5432v1