FA-presentable groups and semigroups

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Languages

Σ : finite set of symbols.

Σ^* : the set of all finite words formed from the symbols in Σ (including the *empty word* ε).

Formal language theory – ''languages'' and abstract models of "machines" that "recognize" languages.

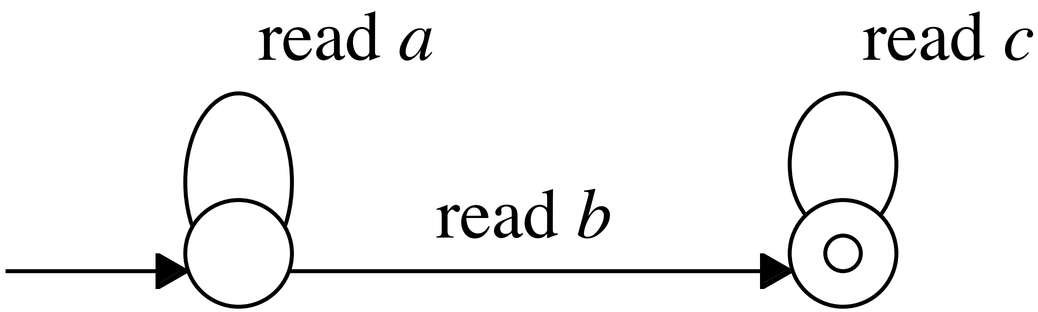
A *language* L is a subset of Σ^* (for some finite set Σ).

To *recognize* L we want an algorithm that decides, given an input $\alpha \in \Sigma^*$, whether or not α is an element of L.

Regular languages

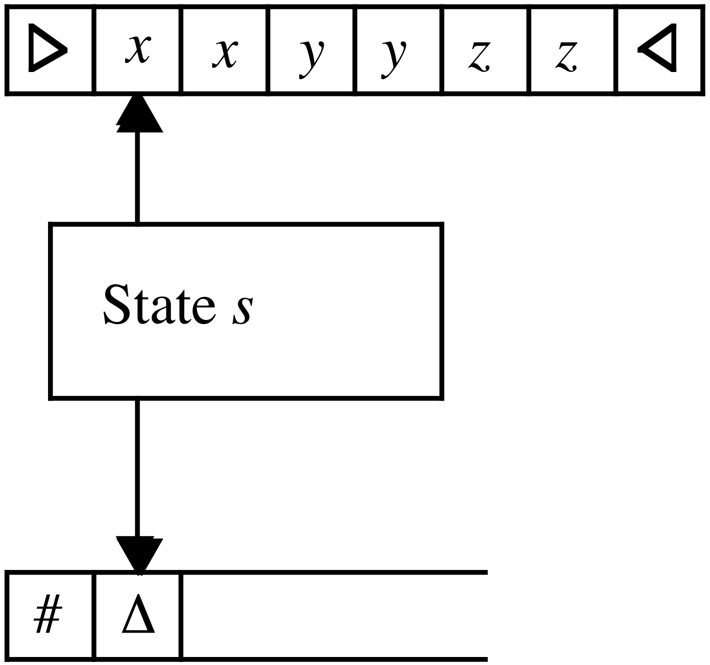
Regular languages are the languages accepted by *finite automata*.

A word α is *accepted* by a finite automaton M if α maps the start state to an accept state. For example, the finite automaton below accepts the language $\{a^n bc^m : n, m \geq 0\}$:



Turing machines

We can also consider a general model of computation such as a *Turing machine*.

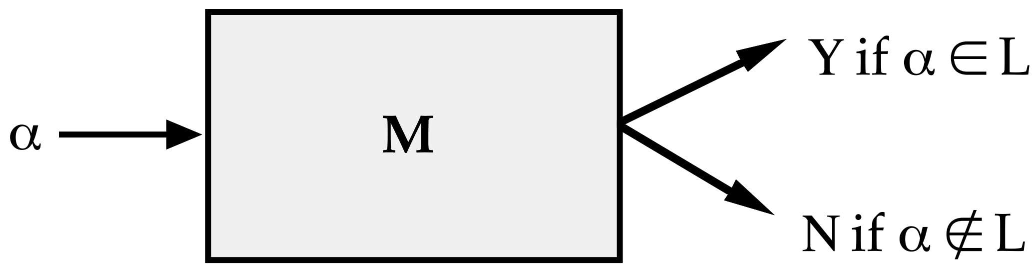


Here we have some memory (in the form of a *work tape*) as well as the input. A Turing machine with a given input will either

(i) terminate (if it enters a halt state); or

(ii) hang (no legal move defined); or

(iii) run indefinitely without terminating.



We will take a *decision-making Turing machine* (one that always terminates and outputs true or false) here (we are considering the class of *recursive languages*).

Structures

A *structure* $\mathcal{A} = (D, R_1, R_2, \ldots, R_n)$ consists of:

- a set D, called the *domain* of $\mathcal{A}$;
- for each i with $1 \leq i \leq n$, there exists $r = r_i \geq 1$ such that $R_i \subseteq D^r$.

A structure $\mathcal{A} = (D, R_1, R_2, \ldots, R_n)$ is said to be *computable* if:

- there is a set of symbols Σ such that $D \subseteq \Sigma^*$ and there is a decision-making Turing machine for D;
- for each R_i there is a decision-making Turing machine that, on input $(\alpha_1, \alpha_2, \ldots, \alpha_r)$, outputs *true* if $\alpha_i \in D$ for each i and $(\alpha_1, \alpha_2, \ldots, \alpha_r) \in R_i$ and outputs *false* otherwise.

FA-presentable structures

A structure $\mathcal{A} = (D, R_1, R_2, \ldots, R_n)$ is said to be *FA-presentable* if:

- there is an alphabet Σ, a language L over Σ and a surjective map $\varphi : L \to D$;
- L is recognized by a finite automaton;
- there is a finite automaton that accepts a pair (α, β) if and only if $\alpha, \beta \in L$ and $\alpha\varphi = \beta\varphi$.
- for each R_i there is a finite automaton that accepts $(\alpha_1, \alpha_2, .., \alpha_r)$ if and only if $\alpha_i \in L$ for all i and $(\alpha_1, \alpha_2, .., \alpha_r) \in R_i$.

Without loss of generality we may assume that φ is bijective.

Nerode & Khoussainov. If $\mathcal{A}$ is an FA-presentable structure then the first-order theory of $\mathcal{A}$ is decidable.

A group can be viewed as a structure $(G, \circ, e, ^{-1})$, where $\circ$ has arity 3, e has arity 1 and $^{-1}$ has arity 2.

Example. Conjugacy in a group is a first-order definable relation:

$$C(a, b) \ := \ (\exists x\colon x^{-1}ax = b).$$

So the conjugacy problem is decidable in a FA-presentable group.

There are not many examples where we have a complete characterization of FA-presentable structures.

Delhomme, Goranko & Knapik. An ordinal α is FA-presentable if and only if $\alpha < \omega^\omega$.

Khoussainov, Nies, Rubin & Stephan. An integral domain is FA-presentable if and only if it is finite.

$\mathcal{B}$: Boolean algebra of all finite and cofinite subsets of **N**.

Khoussainov, Nies, Rubin & Stephan. An infinite Boolean algebra is FA-presentable if and only if it is a direct sum of the form $\mathcal{B}^n$.

FA-presentable groups

Oliver & Thomas. A finitely generated group is FA-presentable if and only if it has an abelian subgroup of finite index.

So, if a finitely generated group is FA-presentable, then it is automatic (but the converse is false).

Nies & Thomas. Every finitely generated subgroup of an FA-presentable group has an abelian subgroup of finite index. Moreover, there exists N such that any such subgroup is a finite extension of $\mathbf{Z}^n$ where $n \leq N$.

Cancellative semigroups

We could consider some naturally occurring classes of semigroups (such as cancellative semigroups).

Cain, Ruskuc, Oliver & Thomas. A finitely generated cancellative semigroup is FA-presentable if and only if it embeds in a (finitely generated) virtually abelian group.

There is an example (due to Alan Cain) of a non-automatic semigroup that is a finitely generated subsemigroup of a virtually abelian group; so a finitely generated cancellative FA-presentable semigroup need not be automatic.

FA-presentable rings

Nies & Thomas. If R is an FA-presentable ring with identity, then every finitely generated subring of R is finite.

This generalizes the earlier result about integral domains. There do exist infinite FA-presentable rings with identity; however, we can show that any FA-presentable division ring is finite.

Nies & Thomas. If R is an FA-presentable commutative ring with identity which is not the direct sum of two non-trivial FA-presentable rings, then R is finite.

Thank you!