On groups with all subgroups subnormal or soluble of bounded derived length

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A subgroup H of a group G is said to be *subnormal* if H is a term of a finite series of G, i.e. if there exists distinct subgroups $H_0, H_1, \dots, H_{n-1}, H_n$ such that

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Question

Is a group with all subgroups subnormal nilpotent?

Dedekind 1897, Baer 1933

All the subgroups of a group G are normal if and only if G is abelian or the direct product of a quaternion group of order 8, an elementary abelian 2-group and an abelian group with all its elements of odd order.

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Roseblade, 1965

Let G be a group in which every subgroup is subnormal of defect at most $n \geq 1$. Then G is nilpotent of class bounded by a function depending only on n.

Heineken and Mohamed, 1968

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Casolo 2001, Smith 2001

A torsion-free group with all subgroups subnormal is nilpotent.

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Let G be a locally (soluble-by-finite) group with all subgroups subnormal or nilpotent. Then G is soluble. Moreover, if G is torsion-free, then it is nilpotent.

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Let G be a locally graded group and suppose that, for some $n \geq 1$, every non-nilpotent subgroup of G is subnormal of defect at most n in G. Then G is soluble. Moreover, if G is torsion-free, then it is nilpotent.

A group is *locally graded* if every non-trivial finitely generated subgroup has a non-trivial finite quotient, e.g. locally (soluble-by-finite) groups and residually finite groups.

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This restriction is made in order to avoid *Tarski groups*, i.e. infinite 2-generator simple groups with all proper non-trivial subgroups of prime order.

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(*iv*) $PSL(3, 3)$;

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Proposition

Let G be a finite non-abelian simple group with all proper subgroups metabelian. Then G is isomorphic to one of the following groups:

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Corollary

Every proper subgroup of a finite minimal simple group has derived length at most 5.

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Let G be an infinite locally graded group with all subgroups soluble of derived length $\leq d$. Then G is soluble of derived length $\leq d$.

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Let G be an infinite locally graded group with all subgroups soluble of derived length $\leq d$. Then G is soluble of derived length $\leq d$.

Zaicev showed that an infinite soluble group of derived length d has a subgroup of derived length d.

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In (ii) one cannot expect that G is an extension of a soluble group by a finite minimal simple group : it suffices to consider the direct product of any abelian group by the symmetric group of degree 5.

Theorem A

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Proof of Theorem A:

By Proposition A, G is either locally soluble, or $G^{(r)}$ is finite for some integer r and G is an extension of a soluble group S by a finite almost minimal simple group.

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Moreover, $G^{(s)}$ is not soluble and every proper subgroup of $G^{(s)}$ is soluble of length at most d.

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Moreover, $G^{(s)}$ is not soluble and every proper subgroup of $G^{(s)}$ is soluble of length at most d. Thus $G^{(s)}$ is finite by Zaicev's result, a contradiction.

Proposition B

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(i) G is soluble of derived length not exceeding a function depending on n and d, or

(ii) $G^{(r)}$ is finite for some integer $r = r(n)$ and G is an extension of a soluble group of derived length at most d by a finite almost minimal simple group.