Symplectic alternating algebras

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1. Introduction.
2. Some general structure theory.
3. Nilpotent symplectic alternating algebras.

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Definition. Let F be a field. A symplectic alternating algebra over F is a triple $(V, (\ ,\ ), \cdot)$ where V is a symplectic vector space over F with respect to a non-degenerate aternating form $(\ ,\ )$ and $\cdot$ is a bilinear and alternating binary operation on V such that

$$(u \cdot v, w) = (v \cdot w, u)$$

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Origin.There is a 1-1 correpondence between SAA's over the field GF(3) and a certain class of powerful 2-Engel groups of exponent 27.

Let L be a SAA. A standard basis for L is a basis $(x_1, y_1, \ldots, x_r, y_r)$ where $(x_i, y_i) = 1$ and $L = (Fx_1 + Fy_1) \oplus_\perp \cdots \oplus_\perp (Fx_r + Fy_r)$

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The map $L^3 \to F,\ (u, v, w) \mapsto (u \cdot v, w)$ is an alternating ternary form and each alternating ternary form defines a unique symplectic alternating algebra. Classifying symplectic alternating algebras of dimension $2r$ over F is then equivalent to finding all the Sp(V) orbits of $\wedge^3 V$, under the natural action, where V is the symplectic vectorspace of dimension $2r$ with non-degenerate alternating form.

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Over the field $\mathbb{Z}_3$ there are 31 algebras of dimension 6 (T, 2008).

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Theorem 4.(Tota, Tortora, T) Let L be a symplectic alternating algebra that is abelian-by-(class c). We then have that L is nilpotent of class at most $2c+1$.

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Lemma 3. Let I and J be ideals of L. Then $Jx \leq I \Leftrightarrow I^{\perp}x \leq J^{\perp}$.

Proposition 5. There exists an ascending chain of isotropic ideals

$$\{0\} = I_0 < I_1 < \cdots < I_{n-1} < I_n$$

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$$\{0\} < I_2 < I_3 < \cdots < I_{n-1} < I_{n-1}^{\perp} < I_{n-2}^{\perp} < \cdots < I_2^{\perp} < L$$

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Presentation We can pick a standard basis $(x_1, y_1, x_2, y_2, \cdots, x_n, y_n)$ such that

$$I_1 = Fx_n,\ I_2 = I_1 + Fx_{n-1},\ \cdots I_n = I_{n-1} + Fx_1,$$
$$I_{n-1}^{\perp} = I_n + Fy_1,\ I_{n-2}^{\perp} = I_{n-1}^{\perp} + Fy_2,\ \cdots, I_0^{\perp} = L = I_1^{\perp} + Fy_n$$

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Here the only triples that are not neccessarily zero are

$$(x_i y_j, y_k) = \alpha_{ijk},\ (y_i y_j, y_k) = \beta_{ijk}\ \ 1 \le i < j < k \le n. \tag{1}$$

Conversely any such presentation (1) gives us a nilpotent SAA with ascending chain $I_1 < I_2 < \cdots < I_n$ as above.

SAA's of maximal class

Theorem 6. Let L be nilpotent SAA of dimension $2n$ and of maximal class. We have for $k = 2, \ldots, n-1$ that

$$Z_{k-1}(L) = L^{2n-1-k}$$

is the unique ideal of dimension k and for $k = n+1, \ldots, 2n-2$ we have that

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Remark. For each $n \geq 4$, there exists a SAA of dimension $2n$ that is of maximal class.