Certain Monomial Characters and Their Subnormal Constituents

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This is a joint work with G. Navarro.

Introduction

Let G be a group. A character $\chi \in \mathrm{Irr}(G)$ is said to be **monomial** if there exist a subgroup $U \subseteq G$ and a linear $\lambda \in \mathrm{Irr}(U)$, such that

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A group G is said to be **monomial** if all its irreducible characters are monomial.

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Thus, supersolvable groups are monomial groups. But this result depends more on the structure of the group than on characters themselves.

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We give a monomiality criterium which also deals with fields of values and degrees of characters.

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Theorem A

Let G be a p-solvable group. Assume that $|\mathbf{N}_G(P) : P|$ is odd, where $P \in \mathrm{Syl}_p(G)$ for some prime p. If $\chi \in \mathrm{Irr}(G)$ has degree not divisible by p and the values of χ are contained in the cyclotomic extension $\mathbb{Q}_{|G|_p}$, then χ is monomial.

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When $p = 2$, we can recover Gow's result from Theorem A.

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B_π Theory

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A $\mathbf{B}_\pi$ **character** of a group G may be thought as an irreducible character of G induced from a π-special character of some subgroup of G. (True in groups of odd order).

Main results

Theorem B

Let G be a p-solvable group. Assume that $|\mathbf{N}_G(P) : P|$ is odd, where $P \in \mathrm{Syl}_p(G)$ for some prime p. If $\chi \in \mathrm{Irr}(G)$ has degree not divisible by p and its values are contained in the cyclotomic extension $\mathbb{Q}_{|G|_p}$, then χ is a B_p character of G.

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Notice that B_p characters with degree not divisible by p are monomial. Thus Theorem B implies Theorem A.

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Corollary C

Let G be a p-solvable group. Suppose that $|\mathbf{N}_G(P) : P|$ is odd, where $P \in \mathrm{Syl}_p(G)$ for some prime p. If $\chi \in \mathrm{Irr}(G)$ has degree not divisible by p and its field of values is contained in $\mathbb{Q}_{|G|_p}$, then every subnormal constituent of χ is monomial.

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Corollary C

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Corollary D

Let G be a p-solvable group. Assume that $|\mathbf{N}_G(P) : P|$ is odd, where $P \in \mathrm{Syl}_p(G)$ for some prime p. The number of irreducible characters which have degree not divisible by p and field of values contained in $\mathbb{Q}_{|G|_p}$ equals the number of orbits under the natural action of $\mathbf{N}_G(P)$ on P/P'.

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The number of such characters can be computed locally.

Thanks for your attention!