On locally finite groups with bounded centralizer chains

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(based on joint work with A. Buturlakin)

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- G is a group and $\mathcal{C} = \{C_G(X) \mid X \subseteq G\}$
- G satisfies the minimal (maximal) condition for centralizers if every descending (ascending) chain in $\mathcal{C}$ is finite.

Remark. Since $C_G(X) < C_G(Y) \iff C_G(C_G(X)) > C_G(C_G(Y))$, minimal and maximal conditions for centralizers are equivalent.

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The c-dimension of a group G is the maximum length of a nested chain of centralizers of subsets in G.

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Remark. If H is a subgroup of G, then c-dimension of H is obviously less or equal to c-dimension of G. But the same is not necessary true for c-dimension of a factor group G/N.

E.I. Khukhro, On solubility of groups with bonded centralizer chains, Glasgow Math. J., 51 (2009), 49-54.

Theorem

If a periodic locally soluble group G has finite c-dimension k, then

1. G is soluble of k-bounded derived length;
2. the factor group $G/F(G)$ by the Hirsch-Plotkin radical $F(G)$ has k-bounded rank; and therefore
3. the factor group $G/F_2(G)$ by the second Hirsch-Plotkin subgroup $F_2(G)$ has an abelian subgroup of finite k-bounded index.

- $F(G)$ is the largest normal locally nilpotent subgroup of G
- $F_2(G)$ is the full inverse image of $F(G/F(G))$ in G.

In Khukhro's paper the following conjecture was presented.

Conjecture (A. Borovik)

Let G be a locally finite group of finite c-dimension k. Then

1. the number of nonabelian simple composition factors of G is finite and k-bounded;
2. ...

Theorem 1 (Buturlakin-V.)

Let G be a locally finite group of finite c-dimension k. Then the number of nonabelian composition factors of G is less than $5k$.

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Remark. It is well-known that every nonabelian locally finite simple group contains a non-soluble finite group. Thus the general assertion can be derived from the corresponding one for finite groups.

- the layer $E(G)$ is the subgroup generated by all subnormal quasi-simple subgroups of G
- the generalized Fitting subgroup $F^*(G)$ of a group G is the product of the Hirsch-Plotkin radical $F(G)$ and the layer $E(G)$.

Conjecture (A. Borovik)

Let G be a locally finite group of finite c-dimension k. Suppose S is the full inverse image of the generalized Fitting subgroup $F^*(G/F(G))$ of the factor group $G/F(G)$. Then

1. ...
2. G/S has an abelian subgroup of finite k-bounded index.

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Compare with

Item (3) of Khukhro's theorem

If a periodic locally soluble group G has finite c-dimension k, then $G/F_2(G)$ has an abelian subgroup of finite k-bounded index.

Let P_1, P_2, $\ldots$, P_n be the nonabelian composition factors of a finite group G. Define s_i, where $1 \leqslant i \leqslant n$,

to be the Lie rank of P_i, if P_i is a group of Lie type,

to be the degree of P_i, if P_i is an alternating group,

to be 1 in other cases.

Put $sr(G) = \sum_{i=1}^{n} s_i$.

Theorem 2 (Buturlakin-V.)

If G is a finite group of finite c-dimension k, then $sr(G)$ is k-bounded.

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Corollary

Suppose that G is a finite group of finite c-dimension k and $\overline{G}$ is the factor group of G by its soluble radical. Then $\overline{G}/E(\overline{G})$ has an abelian subgroup of finite k-bounded index.