Hausdorff dimension in pro-p groups

Amaia Zugadi-Reizabal

University of the Basque Country, Bilbao

Groups St Andrews 2013 in St Andrews
August 5th, 2013

Contents

1. Hausdorff dimension
2. Characterization of p-adic analytic solvable pro-p groups

Contents

1. Hausdorff dimension
2. Characterization of p-adic analytic solvable pro-p groups

How big is a subgroup in the whole group?

What could we do to measure the relative size of a subgroup in the whole group?
The inverse of the index $\frac{1}{|G:H|} = \frac{|H|}{|G|}$ might be a good option, or $\frac{\log|H|}{\log|G|}$ if we think of p-groups.

This is a good choice if the group G is finite, but we find some problems if it is infinite:

- It does not distinguish subgroups of infinite index.
- Intuitively, a subgroup of finite index of an infinite group should have dimension 1.

Abercrombie proposed a way to skip this problem in the case of profinite groups, using the concept of Hausdorff dimension of a metric space.

Hausdorff dimension

Hausdorff dimension was originally defined for metric spaces and it is a sharp tool to detect the 'fractalness' of sets.

Hausdorff dimension in profinite groups

Suppose G is a countably based profinite group i.e. there exists a descending chain $\{G_n\}_{n\in\mathbb{N}}$ of open normal subgroups which form a base of neighbourhoods of the identity (if G is topologically finitely generated, then it is countably based).

Let H be a closed subgroup of G. Then, the Hausdorff dimension of H in G is:

$$\dim_G H = \liminf_{n\to\infty} \frac{\log |HG_n/G_n|}{\log |G/G_n|}.$$

This is precisely the formula that **Abercrombie** and **Barnea-Shalev** proved considering the following metric in G:

$$d(x,y) = \inf\{1/|G : G_n| : x \equiv y \bmod G_n\}.$$

Some easy properties

Let G be a countably based profinite group. Then

- If $H \leq_c G$, $\dim H \in [0,1]$.
- If $H, K \leq_c G$, $K \leq H \Rightarrow \dim K \leq \dim H$.
  If in addition, $|H : K| < \infty \Rightarrow \dim K = \dim H$.
- If G is infinite, open subgroups have Hausdorff dimension 1 and finite subgroups 0.

The spectrum of G

Definition

$\mathrm{Spec}\,(G) = \{\dim_G H : H \leq_c G\} \subseteq [0,1]$ is the *spectrum* of G.

It is useful if we want to measure the ‘complexity’ of the subgroup structure of G.

Theorem (Barnea and Shalev)

Let G be a *p-adic analytic pro-p group* and let d denote the dimension of G as a Lie group, then

$$\mathrm{Spec}\,(G) \subseteq \left\{0, \frac{1}{d}, \frac{2}{d}, \ldots, \frac{d-1}{d}, 1\right\}$$

is finite and contains just rational numbers.

Contents

1 Hausdorff dimension

2 Characterization of p-adic analytic solvable pro-p groups

p-adic analytic pro-p groups

Definition

A *p-adic analytic group* is a group with the structure of an analytic manifold over $\mathbb{Q}_p$ such that group multiplication and inversion are both analytic functions.

Characterizations

Let G be a pro-p group. Then the following are equivalent:

1. G is a p-adic analytic group;
2. G has finite rank;
3. G is finitely generated and virtually uniform;
4. G has polynomial subgroup growth;
5. G is isomorphic to a closed subgroup of $GL_d(\mathbb{Z}_p)$ for some d;
6. G is finitely generated, and no infinite closed subgroup of G has Hausdorff dimension zero.

On the spectrum

Theorem (Barnea and Shalev)

Let G be a *p-adic analytic pro-p group* and let d denote the dimension of G as a Lie group, then

$$\mathrm{Spec}\,(G) \subseteq \left\{0, \frac{1}{d}, \frac{2}{d}, \dots, \frac{d-1}{d}, 1\right\}$$

is finite and contains just rational numbers.

Conjecture (Shalev)

If G is a finitely generated pro-p group such that $\mathrm{Spec}\,(G)$ is finite, then G is p-adic analytic.

First observation: We need to fix the filtration!

The filtration really matters

Example (Klopsch, Z-R)

Let $G = \mathbb{Z}_p \times \mathbb{Z}_p$.

- The spectrum corresponding to the pth power filtration is
$$\{0, 1/2, 1\}.$$
- There exists a filtration $\{G_n\}$ for which
$$\left[\frac{1}{p+1}, \frac{p-1}{p+1}\right] \subseteq \mathrm{Spec}\,(G).$$

We fix the system of neighbourhoods to be the pth power filtration, i.e. when $G_n = G^{p^{n-1}}$.

The characterization

Theorem (Klopsch, Z-R)

Let G be a soluble pro-p group not of finite rank, then $\mathrm{Spec}\,(G)$ contains a non-trivial interval.

Corollary

Let G be a finitely generated soluble pro-p group. Then the following are equivalent

- G is p-adic analytic;
- $\mathrm{Spec}\,(G)$ is finite.

Outline of the proof

Let $\left\{G^{(m)}\right\}_{m\geq 1}$ be the derived series of G and take k the maximum such that $\dim_G G^{(k)} = 1$.

If we write $\dim_G G^{(k+1)} = \eta$, for each $K \leq G^{(k)}$ finitely generated, by a result of Shalev, we have

$$\dim_G K \leq \dim_G G^{(k+1)} = \eta < 1.$$

Fix $\xi \in [\eta, 1]$, we are able to build a series $H_0 \leq H_1 \leq H_2 \leq \ldots$ of finitely generated groups such that

$$\dim_G \langle H_i \rangle = \xi.$$

On the normal spetrum

Definition

The *normal spectrum* is the Hausdorff spectrum restricted to normal subgroups.

Question (Shalev)

Is there a group of infinite normal spectrum?

Example (Klopsch, Z-R)

There exists a pro-p group G whose normal spectrum contains an interval.

Thank you!
Eskerrik asko!