Graphs encoding the generating properties of a finite group

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The generating graph

The generating graph (M. Liebeck & A. Shalev, 1996)

The generating graph $\Gamma(G)$ of a finite group G is the graph defined on the elements of G in such a way that two distinct vertices are connected by an edge if and only if they generate G.

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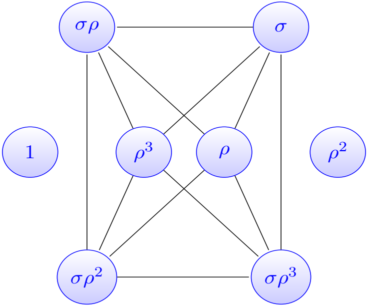

Figure: Generating graph of D_8

The generating graph

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  (see many authors R. M. Guralnick, M. Liebeck, A. Shalev, W. M. Kantor, T. Breuer, A. Lucchini, A. Maróti, C. M. Roney-Dougal, G. P. Nagy, etc)

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- If G is not generated by two elements, then the graph $\Gamma(G)$ is empty.
- The generating graph encodes significant information only when G is a 2-generator group.
- We introduce and investigate a wider family of graphs which encode the generating property of G when G is an arbitrary finite group.

A natural generalization of $\Gamma(G)$

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Let $d(G)$ the smallest cardinality of a generating set of G.

If $a + b < d(G)$, then $\Gamma_{a,b}(G)$ is an empty graph, so in general we implicitly assume $a + b \geq d(G)$.

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The graph $\Gamma^*_{a,b}(G)$

We define the graph $\Gamma^*_{a,b}(G)$ as the graph obtained from $\Gamma_{a,b}(G)$ by deleting the isolated vertices.

Some observations on the graphs $\Gamma_{a,b}(G)$ and $\Gamma^*_{a,b}(G)$

Let $\Phi_G(d) = \{(x_1, \ldots, x_d) \mid G = \langle x_1, \ldots, x_d \rangle\}$ and $\phi_G(d)$ be its cardinality.

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- Let $d = a + b$. If $a \neq b$ then $\Gamma_{a,b}(G)$ and $\Gamma^*_{a,b}(G)$ are bipartite graphs with two parts, one corresponding to the elements of G^a and the other to the elements of G^b.

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  The number of edges of $\Gamma_{a,b}(G)$ (of $\Gamma^*_{a,b}(G)$) is $\phi_G(d)$.

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- If $a = b$, then $\Gamma_{a,a}(G)$ has $|G|^a$ vertices, $\phi_G(a)$ loops and other $(\phi_G(d) - \phi_G(a))/2$ edges connecting two different vertices: the two elements $(g_1, \ldots, g_a, g_{a+1}, \ldots, g_d)$ and $(g_{a+1}, \ldots, g_d, g_1, \ldots, g_a)$ give rise to the same edge in $\Gamma_{a,a}(G)$.

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- For any a, no connected component of $\Gamma^*_{a,a}(G)$ is bipartite.
- If $|G| \geq 3$, then $\Gamma^*_{a,b}(G)$ contains a vertex x of degree 1 if and only if $a = 0$, $b \geq d(G)$ and x is one of the $\phi_G(b)$ leaves of the star $\Gamma^*_{0,b}(G)$.

Connectivity

The swap graph

For a d-generator finite group G, the swap graph $\Sigma_d(G)$ is the graph in which the vertices are the ordered generating d-tuples and in which two vertices $(x_1, \ldots, x_d)$ and $(y_1, \ldots, y_d)$ are adjacent if and only if they differ only by one entry.

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The result above is equivalent to say that "swap conjecture" is satisfied by the 2-generator finite soluble groups.

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Theorem(E.Crestani & A.Lucchini '13, M.Di Summa & A.Lucchini '16)
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Open problem:
to decide whether $\Gamma^*_{a,b}(G)$ is connected when $a + b = d(G)$ and G is unsoluble.

We conjecture that the answer is positive.

Bounding the diameter of $\Gamma^*_{1,1}(G)$ in the soluble case

Theorem (A. Lucchini '17)

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Note that if the derived subgroup of G is nilpotent or has odd order, then G satisfy P($*$).

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Theorem (AC & A. Lucchini '17)

Assume that G is a finite soluble group and that $(x_1, \ldots, x_b)$ and $(y_1, \ldots, y_b)$ are non-isolated vertices of $\Gamma_{a,b}(G)$: if either $a \neq 1$ or G satisfies the property P($*$), then there exists $(z_1, \ldots, z_a) \in G^a$ such that $G = \langle z_1, \ldots, z_a, x_1, \ldots, x_b \rangle = \langle z_1, \ldots, z_a, y_1, \ldots, y_b \rangle$.

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Let G be a finite soluble group and let a and b non-negative integers such that $a + b \geq d(G)$. Then $\operatorname{diam}(\Gamma^*_{a,b}(G)) \leq 4$.

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(1) *Assume $a = b$. If either G has the property P($*$) or $a \neq 1$, then $\mathrm{diam}(\Gamma^*_{a,a}(G)) \leq 2$. Otherwise $\mathrm{diam}(\Gamma^*_{a,a}(G)) \leq 3$.*

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(2) *Assume $a < b$. If either G has the property P($*$) or $a \neq 1$, then $\operatorname{diam}(\Gamma^*_{a,b}(G)) \leq 3$.*

Bounding the diameter of the swap graph

Theorem (AC & A. Lucchini '17)

If G is soluble and has property P($*$) , then the diameter of the swap graph $\Sigma_d(G)$ is at most $2d-1$, whenever $d \geq d(G)$.

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Theorem (AC & A. Lucchini '17)

If a and b are positive integers, then

$$\lim_{p\to\infty} \operatorname{diam}(\Gamma^*_{a,b}(\mathrm{SL}(2,2^p)^{\tau_{a+b}(\mathrm{SL}(2,2^p))})) = \infty.$$

Recovering information on G from the graphs

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We can think that we packaged all the graphs $\Gamma^*_{a,b}(G)$ in a (quite spacious) box but that we did not paid enough attention during this operation and we lost the information to which group G these graphs correspond and the labels a, b.

DO NOT PANIC! a big amount of the lost information can be reconstructed!

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- For a given finite group G, $P_G(s)$ can be determined from $\Lambda^*(G)$.

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$P_G(s)$ is a uniquely determined Dirichlet polynomial, with s a complex variable and satisfying the property that for $t \in \mathbb{N}$ the number $P_G(t)$ coincides with the probability of generating G by t randomly chosen elements.

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- we may deduce whether G is soluble or supersoluble;
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For example we cannot deduce from $P_G(s)$ whether G is nilpotent.

Example

Consider $G_1 = C_6 \times C_3$ and $G_2 = \mathrm{Sym}(3) \times C_3$. It turns out that

$$P_{G_1}(s) = P_{G_2}(s) = \left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{3}{3^s}\right).$$

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On the contrary $\Lambda^*(G)$ encodes enough information to decide whether G is nilpotent.

Theorem (AC & A. Lucchini '17)

Let G be a finite nilpotent group. If H is a finite group and $\Lambda^*(H) = \Lambda^*(G)$, then H is nilpotent.

More properties of G: the order of the Frattini subgroup

Another information that we cannot recover from the knowledge of $|G|$ and $P_G(s)$ is the order of $\mathrm{Frat}(G)$.

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We have that $|G_1| = |G_2| = 20$ and $P_{G_1}(s) = P_{G_2}(s) = \left(1 - \frac{1}{2^s}\right)\left(1 - \frac{5}{5^s}\right)$
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In same crucial steps of the proofs of our results (for example when we recognize the nilpotency and $|\operatorname{Frat}(G)|$ of G) a decisive role is played by the graphs $\Gamma^{*}_{1,t}(G)$.

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We can consider the family $\Lambda_1^{*}(G)$ of the connected components of the graphs $\Gamma^{*}_{1,t}(G)$ for $t \in \mathbb{N}$.

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We can consider the family $\Lambda_1^{*}(G)$ of the connected components of the graphs $\Gamma^{*}_{1,t}(G)$ for $t \in \mathbb{N}$.

Theorem (AC & A. Lucchini '17)

Assume that the family $\Lambda_1^{*}(G)$ is known. We may determine $|G|$, $d(G)$, $P_G(s)$ and $|\mathrm{Frat}(G)|$. Moreover we may recognize whether or not G is soluble, supersoluble, nilpotent.

The following notion was recently introduced by P. Cameron, A. Lucchini and C. Roney-Dougal:

Efficiently generation

A finite group G is efficiently generated if for all $x \in G$, $d_{\{x\}}(G) = d(G)$ implies that $x \in \mathrm{Frat}(G)$, where $d_{\{x\}}(G)$ denotes the smallest cardinality of a set of elements of G generating G together with x.

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Generalizing another definition given by the same authors for 2-generator groups, we say that

Non-zero spread

A finite group G has non-zero spread if (g) is not isolated in the graph $\Gamma_{1,d(G)-1}(G)$ for every $g \neq 1$.

Finite groups with non-zero spread

Theorem (AC & A. Lucchini '17)

Let G be a finite group.

(1) *Assume that the family $\Lambda_1^*(G) = \{\Gamma^*_{1,r-1}(G)\}_{r\in\mathbb{N}}$ is known. We may deduce whether G is or not efficiently generated.*

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(2) *G has non-zero spread if and only if G is efficiently generated and has trivial Frattini subgroup.*

If G is a finite group with non-zero spread, then G has the property that every proper quotient can be generated by $d(G)-1$ elements, but G cannot.

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Let L be a monolithic primitive group and let A be its unique minimal normal subgroup. For each positive integer k, let L^k be the k-fold direct product of L. The crown-based power of L of size k is the subgroup L_k of L^k defined by $L_k = \{(l_1, \dots, l_k) \in L^k \mid l_1 \equiv \dots \equiv l_k \text{ mod } A\}$.

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From results of F. Dalla Volta and A. Lucchini one can deduce:

If G has non-zero spread, then there exist a monolithic primitive group L and a positive integer t such that $G \cong L_t$ and $d(L_{t-1}) < d(L_t)$ (setting $L_0 = L/\operatorname{soc}(L)$).

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Conversely

Theorem (AC & A. Lucchini '17)

Let L be a finite monolithic primitive group and $t \in \mathbb{N}$. Assume that $G \cong L_t$ and $d(L_{t-1}) < d(L_t)$, then G has non-zero spread, except possibly when $t = 1$ and $d(L) = 2$.

Thank you!