Generating pairs for the Fischer's group Fi_{23}

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Abstract

A group G is called (l, m, n)-generated, if it is a quotient of the triangle group $T(l, m, n) = \langle x, y, z \mid x^l = y^m = z^n = xyz = 1 \rangle$. Moori [13] posed the question of finding all the triples (l, m, n) such that non-abelian finite simple groups are (l, m, n)- generated. In the present article, we answer this question for the Fischer sporadic simple group Fi_{23}. In particular, we compute (p, q, r)-generations for the Fischer group Fi_{23}, where p, q and r are prime divisors of $|Fi_{23}|$.

- Group generations have played a significant role in solving problems in diverse areas of mathematics such as topology, geometry and number theory.
- Generation of a group by its suitable subsets have been the subject of research since the origions of geoup theory.

- A group G is said to be (l, m, n)*-generated* if $G = \langle x, y \rangle$, with $o(x) = l$, $o(y) = m$ and $o(xy) = n$.

- In such case, G is a quotient group of the Von Dyck group $D(l, m, n)$, and therefore it is also $(\pi(l), \pi(m), \pi(n))$-generated for any $\pi \in S_3$. Thus we may assume throughout that $l \leq m \leq n$.

- Further, we emphasize that attention may be restricted to (p, q, r)-generations where p, q, r are primes. Indeed, (l, m, n)-generation follows from (p, q, r)-generation provided $p = l^\alpha$, $q = m^\beta$, $r = n^\gamma$ for some $\alpha, \beta, \gamma \in \mathbb{Z}$.

- Initially, the study of (l, m, n)-generations of a group G had deep connections to the topological problem of determining the least genus of an orientable surface on which G admits an effective, orientation-preserving, conformal action.
- In MOORI [13], such investigations were extended well beyond the “minimum genus problem" to all possible (l, m, n)-generations, assuming G to be finite and non-abelian simple.
- Generational results of this type have since proved to be quite useful and interesting.

- Groups that can be generated by an involution and an element of order 3 are said to $(2,3)$-*generated*, and such generations have been of particular interest to combinatorists and group theorists.

- Any group generated by an involution and an element of order 3 is a quotient group of $PSL(2,\mathbb{Z})$.

- Connections with Hurwitz groups, regular maps, Beauville surfaces and structures provide additional motivation for the study of these groups. (Recall that a *Hurwitz group* is one that can be $(2,3,7)$-generated.)

- If a simple group G is (l, m, n)-generated, then by CONDER [5] either $G \cong A_5$ or $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$.
- Moori in [Nova Journal of Algebra and Geometry 2 (1993), 277-285] posed the following problem.
- **Problem**

  Given a non-abelian finite simple group G with l, m and n dividing $|G|$ such that $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$. Is G (l, m, n)-generated?

Many researchers have answered this question since then:

- In a series of articles Prof. Moori (with his research team at Pietermaritzburg):

  HS, McL, J_i $(1 \leq i \leq 4)$ Co_2, Co_3 and Fi_{22}

- Prof. Darafsheh and Prof. Ashrafi (with their research teams in Iran):

  Co_1, Th, $O'N$, Ly, He.

- Further, in collaborations with Prof. Moori and Prof. Woldar, we investigated the groups:

  Fi_{23}, Fi'_{24}, The Baby Monster group $\mathbb{B}$

- In the present talk, we investigate (p, q, r)-generations for the Fischer group Fi_{23}. Since $(2, 3, 3)$- and $(2, 3, 5)$-generated groups are quotients of A_4 and A_5 respectively, we need only to consider here the cases when $r = 7, 11, 13, 17, 23$.

General Theory and Techniques

Throughout this article we use the same notation and terminology as can be found in [1, 2, 8, 10, 14]. In particular, for a finite group G with conjugacy classes C_1, C_2, C_3, we denote the corresponding structure constant of G by $\Delta(G) = \Delta_G(C_1, C_2, C_3)$. Observe that $\Delta(G)$ is nothing more than the cardinality of the set $\Omega = \{(x, y) | xy = z\}$ where $x \in C_1$, $y \in C_2$ and z is a fixed representative in the conjugacy class C_3. It is well known that the value of $\Delta(G)$ can be computed from the character table of G (e.g., see [11, p.45]) via the formula

$$\Delta_G(C_1, C_2, C_3) = \frac{|C_1||C_2|}{|G|} \sum_{i=1}^{m} \frac{\chi_i(g_1)\chi_i(g_2)\cdots\chi_i(g_{k-1})\overline{\chi_i(g_k)}}{[\chi_i(1_G)]^{k-2}}$$

where $\chi_1, \chi_2, \cdots, \chi_m$ are the irreducible complex characters of G, and the bar denotes complex conjugation.

We denote by $\Delta^*(G) = \Delta^*_G(C_1, C_2, C_3)$ the number of distinct ordered pairs $(x, y) \in \Omega$ such that $G = \langle x, y\rangle$. Clearly, if $\Delta^*(G) > 0$ then G is (l, m, n)-generated where l, m, n are the respective orders of elements from C_1, C_2, C_3. In this instance we shall also say that G is (C_1, C_2, C_3)*-generated* and we shall refer to (C_1, C_2, C_3) as a *generating triple* for G.

Further, if H is a subgroup of G containing the fixed element $z \in C_3$ above, we denote by $\Sigma(H) = \Sigma_H(C_1, C_2, C_3)$ the total number of distinct ordered pairs $(x, y) \in \Omega$ such that $\langle x, y\rangle \leq H$. The value of $\Sigma_H(C_1, C_2, C_3)$ is obtained as the sum of all structure constants $\Delta_H(c_1, c_2, c_3)$ where the c_i are conjugacy classes of H that fuse to C_i in G, i.e., $c_i \subseteq H \cap C_i$. The number of pairs $(x, y) \in \Omega$ generating a subgroup H of G will be denoted by $\Sigma^*(H) = \Sigma^*_H(C_1, C_2, C_3)$, and the centralizer of a representative of the conjugacy class C by $C_G(C)$.

A general conjugacy class of a proper subgroup H of G whose elements are of order n will be denoted by nx, reserving the notation nX for the case where $H = G$. The number of conjugates of a given subgroup H of G containing a fixed element g is given by $\pi(g)$, where π is the permutation character corresponding to the action of G on the cosets of H, i.e., π is the induced character $(1_H)^G$ ([11]). As the stabilizer of H in this action is clearly $N_G(H)$, in many cases one can more easily compute the value $\pi(g)$ from the fusion map from $N_G(H)$ into G in conjunction with Theorem 4.1 below. We emphasize that this is an especially useful strategy when the decomposition of π into irreducible characters is not known explicitly.

Thus, in order to compute $\Delta^*(G)$, we need the character tables of G and character tables of $M_1, M_2, \ldots, M_t$ together with information on $M_i \cap M_j$. However, amongst the maximal subgroups M_j containing z, there may be conjugate subgroups. In such situation the following theorem is very helpful.

Theorem

(GANIEF & MOORI [10]) *Let G be a finite group and let H be a subgroup of G containing a fixed element g such that $gcd(o(g), |N_G(H) : H|) = 1$. Then the number of conjugates of H containing z is given by*

$$\pi(g) = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_{N_G(H)}(g_i)|}$$

where π is the permutation character corresponding to the action of G on the cosets of H, and $g_1, g_2, \ldots, g_m$ are representatives of the $N_G(H)$-conjugacy classes that fuse to the G-class containing g.

Non-Generation

Below we provide some very useful techniques for establishing non-generation.

Lemma

(CONDER, WILSON, & WOLDAR [6]) *Let G be a finite centerless group and suppose lX, mY, nZ are G-conjugacy classes for which*

$$\Delta^*(G) = \Delta^*_G(lX, mY, nZ) < |C_G(nZ)|.$$

Then $\Delta^*(G) = 0$ *and therefore G is not* (lX, mY, nZ)*-generated.*

Lemma

(CONDER *[5]*) *Suppose a and b are permutations of N points such that a has* λ_u *cycles of length u* $(1 \leq u \leq l)$ *and b has* μ_v *cycles of length v* $(1 \leq v \leq m)$ *and their product ab is an involution having k transpositions and* $N - 2k$ *fixed points. If a and b generate a transitive group on these N points, then there exists a non-negative integer* α *such that*

$$k = 2\alpha - 2 + \sum_{1 \leq u \leq l} \lambda_u + \sum_{1 \leq v \leq m} \mu_v.$$

Definition

A group G is called a 3-transposition group *if it is generated by a conjugacy class D of involutions in G such that $o(de) \leq 3$ for all d and e in D. The conjugacy class D is called a class of conjugate* 3*-transpositions.*

- *Fischer* introduced and investigated 3-transposition groups. He classified all finite 3-transposition groups with no non-trivial normal soluble subgroups.
- In the process of classifying the 3-transposition groups, Fischer discovered three new groups Fi_{22}, Fi_{23} and Fi_{24} with 3510, 31671 and 306936 transpositions respectively.
- Of these, the first two groups are simple, while the third contains a simple normal subgroup Fi_{24}' of index 2 (consisting of the products of evenly many transpositions)

(p,q,r)-Generations of Fi_{23}

- The Fischer's sporadic group Fi_{23} has order

  $4089470473293004800 = 2^{18}.3^{13}.5^{2}.7.11.13.17.23 \ \approx 4\times 10^{18}$

- The group Fi_{23} has 94 conjugacy classes of its elements in total including three involution classes and four classes of elements of order 3, namely $2A, 2B, 2C, 3A, 3B, 3C$ and $3D$ as represented in $\mathbb{ATLAS}$ [7].
- Kleidman, Parker and Wilson [12] classified all the maximal subgroups of Fi_{23}. There are 14 conjugacy classes of maximal subgroups of Fi_{23}.

$(2,3,11)$-Generations of Fi_{23}

In order to investigate $(p,q,11)$-generations of Fi_{23} we require knowledge of all the its maximal subgroups with order divisible by 11. They are, up to isomorphisms,

$$2{\cdot}Fi_{22}, \quad 2^2{\cdot}U_6(2).2, \quad 2^{11}{\cdot}M_{23}, \quad S_{12}, \quad L_2(23)$$

Lemma

The group Fi_{23} is $(2X,3Y,11A)$-generated, for $X \in \{A,B,C\}$ and $Y \in \{A,B,C,D\}$, if and only if the ordered pair $(X,Y)=(C,D)$.

Proof:

As Fi_{23} has unique class of elements of order 11, we have 12 triples of classes to consider in this case.

Set $T=\{(2A, 3Y, 11A), (2B, 3A, 11A), (2B, 3B, 11A), (2C, 3A, 11A), (2$
For any triple $(2X, 3Y, 11A) \in T$, non-generation follows at once since $\Delta_{Fi_{23}}(2X, 3Y, 11A) = 0$ in those case. Further $(2B, 3C, 11A)$ is not a generating triple for Fi_{23} since

$$\Delta_{Fi_{23}}(2B, 3C, 11A) = 11 < 44 = |C_{Fi_{23}}(11A)|.$$

Next, we consider the case $(2B, 3D, 11A)$.

Case $(2B, 3D, 11A)$

Let $L \cong M_{12}$ be contained in the conjugacy class of subgroups with non-empty intersection with the classes $2B, 3D$ and $11A$. Observe that $N_{Fi_{23}}(L) = C_2 \times M_{12}$.

Let $z \in L$ be a fixed element of order 11. Then the fusion map of L into Fi_{23} yields

$$2a \to 2B, \qquad 3a \to 3D, \qquad 11a \to 11A, \qquad 11b \to 11B.$$

Since $|C_{C_2 \times M_{12}}(z)| = 22$ and $|C_{Fi_{23}}(z)| = 44$, it follows that z is contained in exactly 4 conjugates of M_{12}.

Further, note that no maximal subgroup of L and hence no proper subgroup of L is $(2B, 3D, 11A)$-generated.

We calculate $\Sigma_{M_{12}}(2B, 3D, 11A) = 11 = \Sigma_{C_2 \times M_{12}}(2B, 3D, 11A)$. Therefore

$$\begin{aligned} \Delta^*_{Fi_{23}}(2B, 3D, 11A) &\leq \Delta_{Fi_{23}}(2B, 3D, 11A) - 4\,\Sigma_{M_{12}}(2B, 3D, 11A) \\ &= 44 - 4(11) = 0, \end{aligned}$$

showing that Fi_{23} is not $(2B, 3D, 11A)$-generated.

Next we examine the triple $(2C, 3C, 11A)$. For this we consider the transitive action of the group Fi_{23} on the cosets of $2.Fi_{22}$ with permutation character

$$\pi = 1a + 782a + 30888a$$

(see [7]). Recall that the value of $\pi(g)$, $g \in Fi_{23}$, is the number of cosets of Fi_{23} fixed by g. Set $N = |Fi_{23} : 2.Fi_{22}| = 31671$. Referring to Lemma 4.3, we have

$$\begin{aligned} \lambda_3 &= \tfrac{N-135}{3} = 10512 \\ \mu_{11} &= \tfrac{N-2}{11} = 2879 \\ k &= \tfrac{N-183}{2} = 15744 \end{aligned}$$

from which we get a contradiction since $\alpha = \frac{2355}{2} \notin \mathbb{Z}$. Thus Fi_{23} is not $(2C, 3C, 11A)$-generated.

Case $(2C, 3D, 11A)$

- Finally, we consider the triple $(2C, 3D, 11A)$. We calculate the structure constant $\Delta_{Fi_{23}}(2C, 3D, 11A) = 11616$.

- The maximal subgroups of Fi_{23} with order divisible by 11, up to automorphisms, are $2.Fi_{22}$, $2^2{\cdot}U_6(2).2$, $2^{11}{\cdot}M_{23}$, S_{12} and $L_2(23)$. However, the maximal subgroups $2^2{\cdot}U_6(2).2$, and $2^{11}{\cdot}M_{23}$ does not meet the Fi_{23}-conjugacy class $3D$. That is, $3D \cap 2^2{\cdot}U_6(2).2 = \emptyset = 2^{11}{\cdot}M_{23} \cap 3D$.

- Further, a fixed element z of order 11 is contained in two conjugate copies of $2.Fi_{22}$, four copies of S_{12} and 20 copies of $L_2(23)$. By looking at the fusion maps from three maximal subgroups into the Fischer group Fi_{23}, we calculate

$$
\begin{aligned}
\Delta^*_{Fi_{23}}(2C, 3D, 11A) &\geq \Delta(Fi_{23}) - 2\,\Sigma(2.Fi_{22}) \\
&\quad -4\,\Sigma(S_{12}) - 20\,\Sigma(L_2(23)) \\
&= 11616 - 2(1980) - 4(110) - 20(22) = 6776,
\end{aligned}
$$

and generation of Fi_{23} follows by the triple $(2C, 3D, 11A)$. This completes the proof. ■

By using similar techniques, we compute generating pairs for the Fischer group Fi_{23}. We sumarize our results in the form following theorem:

Theorem

The Fischer group Fi_{23} is (p, q, r)-generated for all $p, q, r \in \{2, 3, 5.7, 11, 17, 23\}$ with $p < q < r$, except when $(p, q, r) = (2, 3, 5)$ or $(p, q, r) = (2, 3, 7)$

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