Descendant-homogeneous digraphs with property Z

Daniela Amato

University of Brasília

August 4, 2017

Notation and Terminology

- *Digraph* D: a pair (D, E), set D of vertices and set $E \subseteq D \times D$ of (directed) edges.

Notation and Terminology

- *Digraph* D: a pair (D, E), set D of vertices and set $E \subseteq D \times D$ of (directed) edges.
- Our digraphs are:
  (a) infinite: set of vertices is infinite
  (b) connected: underlying graph is connected.
  (c) transitive: $\mathrm{Aut}(D)$ is transitive on the set of vertices.

Notation and Terminology

- *Digraph* D: a pair (D, E), set D of vertices and set $E \subseteq D \times D$ of (directed) edges.
- Our digraphs are:
  (a) infinite: set of vertices is infinite
  (b) connected: underlying graph is connected.
  (c) transitive: $\mathrm{Aut}(D)$ is transitive on the set of vertices.
- If $a \in D$, the cardinality of the set of successors $\{b \in D \mid (a, b) \in E\}$ is the *out-valency* of a.
  The cardinality of the set of predecessors $\{c \in D \mid (c, a) \in E\}$ is the *in-valency*.

Notation and Terminology

- *Digraph* D: a pair (D, E), set D of vertices and set $E \subseteq D \times D$ of (directed) edges.
- Our digraphs are:
  (a) infinite: set of vertices is infinite
  (b) connected: underlying graph is connected.
  (c) transitive: $\mathrm{Aut}(D)$ is transitive on the set of vertices.
- If $a \in D$, the cardinality of the set of successors $\{b \in D \mid (a, b) \in E\}$ is the *out-valency* of a.
  The cardinality of the set of predecessors $\{c \in D \mid (c, a) \in E\}$ is the *in-valency*.
- *s-arc*: sequence $u_0 u_1 \ldots u_s$, (u_i, u_{i+1}) an edge, $u_{i-1} \neq u_{i+1}$.

Notation and Terminology

- *Digraph* D: a pair (D, E), set D of vertices and set $E \subseteq D \times D$ of (directed) edges.
- Our digraphs are:
  (a) infinite: set of vertices is infinite
  (b) connected: underlying graph is connected.
  (c) transitive: $\mathrm{Aut}(D)$ is transitive on the set of vertices.
- If $a \in D$, the cardinality of the set of successors $\{b \in D \mid (a, b) \in E\}$ is the *out-valency* of a.
  The cardinality of the set of predecessors $\{c \in D \mid (c, a) \in E\}$ is the *in-valency*.
- s-*arc*: sequence $u_0 u_1 \ldots u_s$, (u_i, u_{i+1}) an edge, $u_{i-1} \neq u_{i+1}$.
- D is s-*arc-transitive* if $\mathrm{Aut}(D)$ is transitive on the set of s-arcs.

Notation and Terminology

- *Digraph* D: a pair (D, E), set D of vertices and set $E \subseteq D \times D$ of (directed) edges.
- Our digraphs are:
  (a) infinite: set of vertices is infinite
  (b) connected: underlying graph is connected.
  (c) transitive: $\mathrm{Aut}(D)$ is transitive on the set of vertices.
- If $a \in D$, the cardinality of the set of successors $\{b \in D \mid (a, b) \in E\}$ is the *out-valency* of a.
  The cardinality of the set of predecessors $\{c \in D \mid (c, a) \in E\}$ is the *in-valency*.
- s-*arc*: sequence $u_0 u_1 \ldots u_s$, (u_i, u_{i+1}) an edge, $u_{i-1} \neq u_{i+1}$.
- D is s-*arc-transitive* if $\mathrm{Aut}(D)$ is transitive on the set of s-arcs.
- *highly arc-transitive*: s-arc transitive for all s.

Notation and Terminology

- *Digraph* D: a pair (D, E), set D of vertices and set $E \subseteq D \times D$ of (directed) edges.
- Our digraphs are:
  (a) infinite: set of vertices is infinite
  (b) connected: underlying graph is connected.
  (c) transitive: $\mathrm{Aut}(D)$ is transitive on the set of vertices.
- If $a \in D$, the cardinality of the set of successors $\{b \in D \mid (a, b) \in E\}$ is the *out-valency* of a.
  The cardinality of the set of predecessors $\{c \in D \mid (c, a) \in E\}$ is the *in-valency*.
- s-*arc*: sequence $u_0 u_1 \ldots u_s$, (u_i, u_{i+1}) an edge, $u_{i-1} \neq u_{i+1}$.
- D is s-*arc-transitive* if $\mathrm{Aut}(D)$ is transitive on the set of s-arcs.
- *highly arc-transitive*: s-arc transitive for all s.
- Example: infinite regular directed tree.

Notation and Terminology

- *Digraph* D: a pair (D, E), set D of vertices and set $E \subseteq D \times D$ of (directed) edges.
- Our digraphs are:
  (a) infinite: set of vertices is infinite
  (b) connected: underlying graph is connected.
  (c) transitive: $\mathrm{Aut}(D)$ is transitive on the set of vertices.
- If $a \in D$, the cardinality of the set of successors $\{b \in D \mid (a, b) \in E\}$ is the *out-valency* of a.
  The cardinality of the set of predecessors $\{c \in D \mid (c, a) \in E\}$ is the *in-valency*.
- s-*arc*: sequence $u_0 u_1 \ldots u_s$, (u_i, u_{i+1}) an edge, $u_{i-1} \neq u_{i+1}$.
- D is s-*arc-transitive* if $\mathrm{Aut}(D)$ is transitive on the set of s-arcs.
- *highly arc-transitive*: s-arc transitive for all s.
- Example: infinite regular directed tree.
- We say that D has *property* Z if there is a digraph homomorphism from D onto Z.

Descendant-homogeneous digraphs

Descendant sets

The descendant set $\mathrm{desc}(u)$ of a vertex u in D is the set of all vertices which can be reached from u by an s-arc, for some $s \geq 0$.

If D is transitive, then all subdigraphs $\mathrm{desc}(u)$ $(u \in D)$ are isomorphic to some fixed rooted digraph Γ. Refer to this as 'the descendant set for D'.

Descendant-homogeneous digraphs

Descendant sets

The descendant set $\text{desc}(u)$ of a vertex u in D is the set of all vertices which can be reached from u by an s-arc, for some $s \geq 0$.

If D is transitive, then all subdigraphs $\text{desc}(u)$ $(u \in D)$ are isomorphic to some fixed rooted digraph Γ. Refer to this as 'the descendant set for D'.

Finitely generated subdigraphs

We say that a subdigraph of a digraph is finitely generated if it is a union of finitely many descendant sets. That is, it is of the form $\text{desc}(X) = \cup_{x \in X} \text{desc}(x)$.

Descendant-homogeneous digraphs

Descendant sets

The descendant set $\mathrm{desc}(u)$ of a vertex u in D is the set of all vertices which can be reached from u by an s-arc, for some $s \geq 0$.

If D is transitive, then all subdigraphs $\mathrm{desc}(u)$ $(u \in D)$ are isomorphic to some fixed rooted digraph Γ. Refer to this as 'the descendant set for D'.

Finitely generated subdigraphs

We say that a subdigraph of a digraph is finitely generated if it is a union of finitely many descendant sets. That is, it is of the form $\mathrm{desc}(X) = \cup_{x \in X} \mathrm{desc}(x)$.

D is descendant-homogeneous if

D is transitive and any isomorphism between finitely generated subdigraphs extends to an automorphism of D

Descendant-homogeneous digraphs

Theorem (DA and John Truss, 2011)

An infinite regular tree of out-valency 1 is descendant-homogeneous.

However,

An infinite regular tree of out-valency > 1 is NOT descendant-homogeneous.

Descendant-homogeneous digraphs

Motivation

Homogeneous graphs (digraphs)

A graph (digraph) is homogeneous if it is countable and any isomorphism between finite subgraphs (subdigraphs) extends to an automorphism.

The class of infinite homogeneous graphs is classified by Lachlan and Woodrow (1980). The class of infinite homogeneous digraphs is classified by Cherlin (1998).

An infinite highly arc transitive digraph (1997)

David Evans constructs an infinite highly arc transitive digraph D such that the descendant set of D is a q-valent directed tree.

We noted that his digraph had an **additional property**, analogous to homogeneity, which we then called *descendant-homogeneity*

Descendant-homogeneous digraphs

Theorem (DA and John Truss, 2011)

There are infinitely many pairwise non-isomorphic descendant-homogeneous digraphs whose descendant sets are rooted q-valent trees where $1 < q < \infty$.

Theorem (DA, David Evans and John Truss, 2011)

The digraphs constructed are the only descendant-homogeneous digraphs in which the descendant set are rooted q-valent trees where $1 < q < \infty$.

The above digraphs do not have property Z and are imprimitive.

Descendant-homogeneous digraphs with property Z

More examples: (DA and John Truss, 2011)

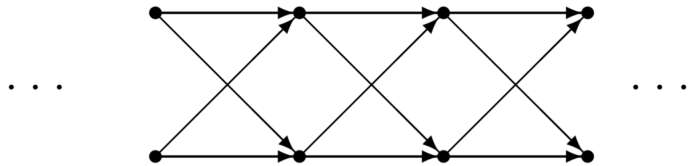
(1) Infinite regular trees of out-valency 1.

Descendant-homogeneous digraphs with property Z

More examples: (DA and John Truss, 2011)

(1) Infinite regular trees of out-valency 1.

(2) .

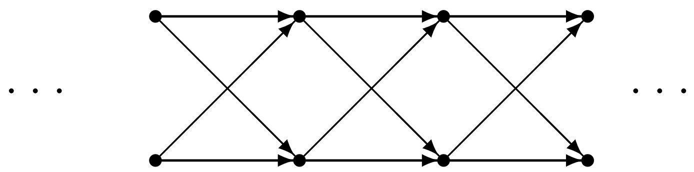


Descendant-homogeneous digraphs with property Z

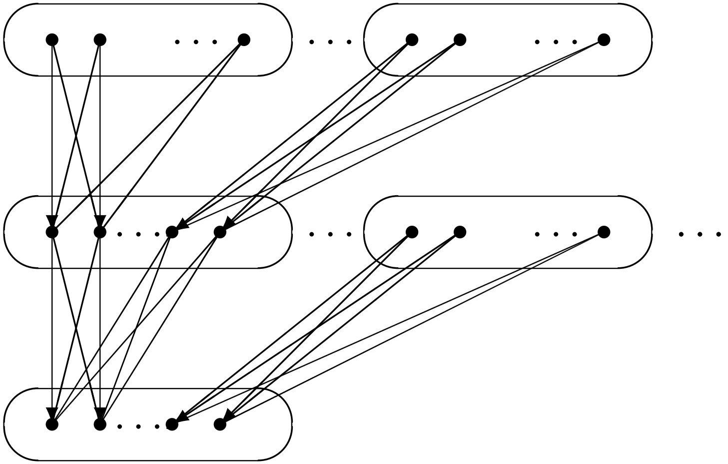
More examples: (DA and John Truss, 2011)

(1) Infinite regular trees of out-valency 1.

(2) .



(3) A combination of (1) and (2).



Descendant-homogeneous digraphs with property Z

All our examples are highly arc transitive and have a homomorphism onto an infinite regular directed tree of out-valency 1.

Lemma

Let D be an infinite, transitive digraph with property Z. Then there is an equivalence relation $\sim$ on the set of vertices of D (preserved by $\mathrm{Aut}(D)$) such that the quotient digraph $D/\sim$ is an infinite regular directed tree of out-valency 1.

.

Question: Is it possible to describe the descendant-homogeneous digraphs which have property Z and are highly arc transitive?

Distance transitive digraphs

Distance transitive digraph

A digraph D is (directed)-distance transitive if for every $s \geq 0$, $\mathrm{Aut}(D)$ is transitive on pairs (u, v) for which there is an s-arc from u to v, but no t-arc for $t < s$.

Note that this implies vertex and edge transitivity but is weaker than being highly arc transitive.

Let D be an infinite, distance transitive digraph of finite out-valency $m > 0$ such that either D has infinite in-valency, or has no directed cycles.

Distance transitive digraphs

Let D be an infinite, distance transitive digraph of finite out-valency $m > 0$ such that either D has infinite in-valency, or has no directed cycles. Let Γ be its descendant set.

The digraph Γ has the following properties

- P0 : $\Gamma = \Gamma(\alpha)$ is a rooted digraph with finite out-valency $m > 0$ and $\Gamma^s(\alpha) \cap \Gamma^t(\alpha) = \varnothing$ whenever $s \neq t$.
- P1 : $\Gamma(u) \cong \Gamma$ for all $u \in \Gamma$.
- P2 : For $i \in \mathbb{N}$ the automorphism group $\mathrm{Aut}(\Gamma)$ is transitive on Γ^i.

For $i \geq 1$, let r_i denote the in-valency of a vertex in Γ^i.
Hence $1 = r_1 \leq r_2 \leq \ldots \leq r_n \leq \ldots$ is an infinite non-decreasing sequence of natural numbers less or equal to m. Hence there is a least natural number $N := N(\Gamma)$ such that $r_j = r_N$ for $j \geq N$.

Distance transitive digraphs

Theorem

Let Γ *be a digraph satisfying* P0 *to* P2*. Then*

- *there is an equivalence relation* ρ *on* Γ *(refining the 'layering' of* Γ*) such that* Γ/ρ *is a (rooted) tree;*
- *there are, up to isomorphism, only countably many such* Γ*. (Joint with David Evans, 2015).*

Recent results

Let D be an infinite, distance transitive, descendant-homogeneous digraph with finite out-valency $m > 0$ and infinite in-valency (or no directed cycles).

- D is imprimitive.

Assume, moreover, that D has property Z.

- $r_N = m$ and $|\Gamma^i| = |\Gamma^{N-1}|$ for all $i \geq N-1$
- If $N(\Gamma) = 1$ then D is an infinite regular tree of out-valency 1.
- If $N(\Gamma) = 2$ then D is isomorphic to the examples (3).
- Corollary: if $N(\Gamma) \in \{1, 2\}$ then D is highly arc transitive.

Recent results

What about examples with $N(\Gamma) \geq 3$?

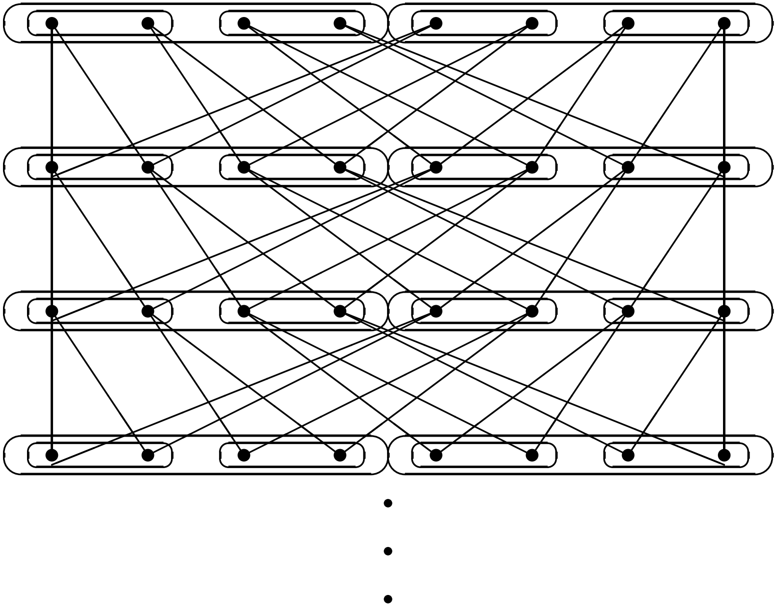
- For each $N \geq 3$, we construct a highly arc transitive digraph D_N of finite out-valency and with $N(\Gamma) = N$.
- D_N has a homomorphism onto an infinite regular tree T of out-valency 1 and in-valency L.

Recent results

What about examples with $N(\Gamma) \geq 3$?

- For each $N \geq 3$, we construct a highly arc transitive digraph D_N of finite out-valency and with $N(\Gamma) = N$.
- D_N has a homomorphism onto an infinite regular tree T of out-valency 1 and in-valency L.
- However, D_N is NOT descendant-homogeneous.

Recent results



References

- DA, Descendant-homogeneous digraphs with property Z, in preparation.
- DA and John K. Truss, Descendant-homogeneous digraphs, Journal of Combinatorial Theory, Series A 31 (2011), 247–283.
- Peter J. Cameron, Cheryl E. Praeger and Nicholas C. Wormald, Infinite highly arc transitive digraphs and universal covering digraphs, *Combinatorica* **13** (1993), 377–396.
- David M. Evans, An infinite highly arc-transitive digraph, *Europ. J. Combin.* **18** (1997), 281–286
- A.H. Lachlan, R.E.Woodrow, Countable ultrahomogeneous undirected graphs. Transactions of the American Mathematical Society 262 (1980) 51–94.
- G. Cherlin, The Classification of Countable Homogeneous Graphs and Countable Homogeneous n-Tournaments, in: Memoirs of the American Mathematical Society, vol.621, 1998.

I would like to thank FAPDF for the travel grant.