Computing Normalisers in Permutation Groups

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Supervised by Dr Chris Jefferson and Dr Colva Roney-Dougal

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Digression: Base

Definition (Base points)

$G \leq Sym(\Omega)$. A **base** of G is $B = [\beta_1, \beta_2, \ldots, \beta_k] \in \Omega^k$ such that $G_{\beta_1,\beta_2,\ldots,\beta_k} = 1$.

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$G = \langle (1,2,3),(4,5) \rangle$. $G_1 = \{(),(4,5)\}$. $G_{1,4} = \{()\}$. So $[1,4]$ is a base of G.

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Lemma (Uniqueness of base image)

The base image B^g of g uniquely determines $g \in G$.

Finding Normalisers by Backtrack search

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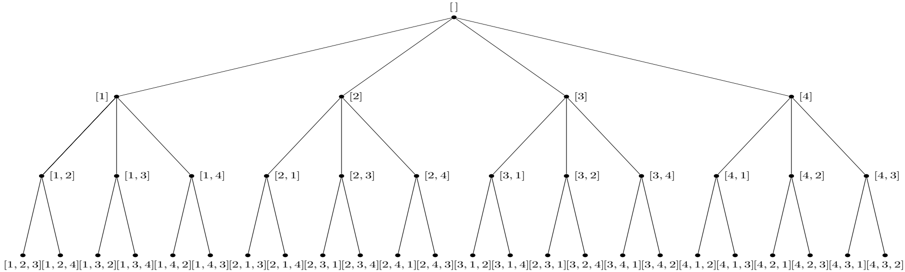

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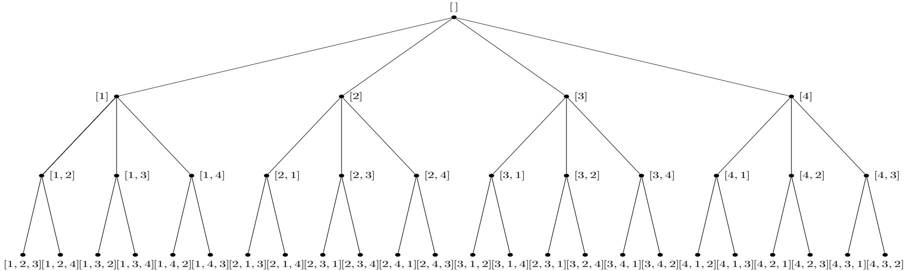

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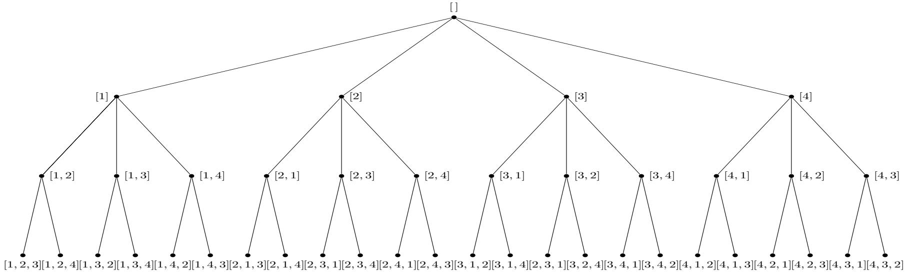

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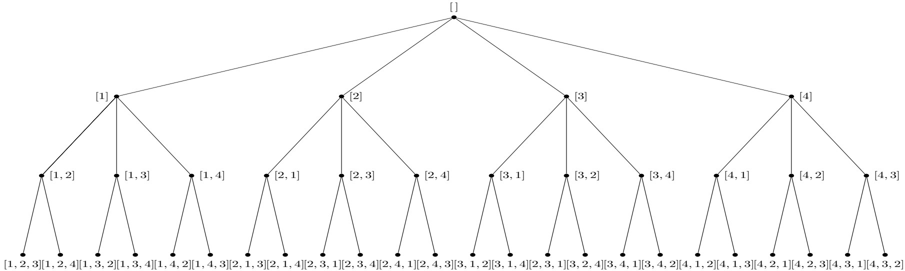

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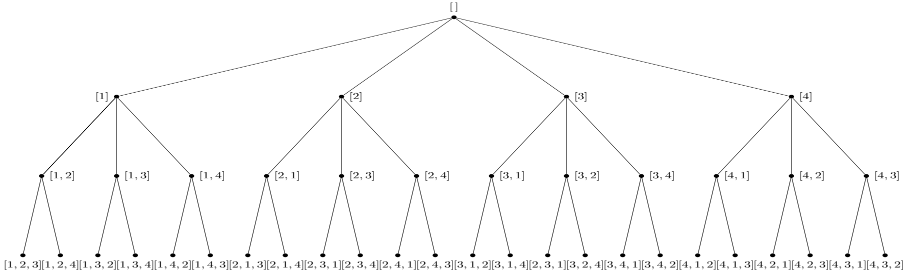

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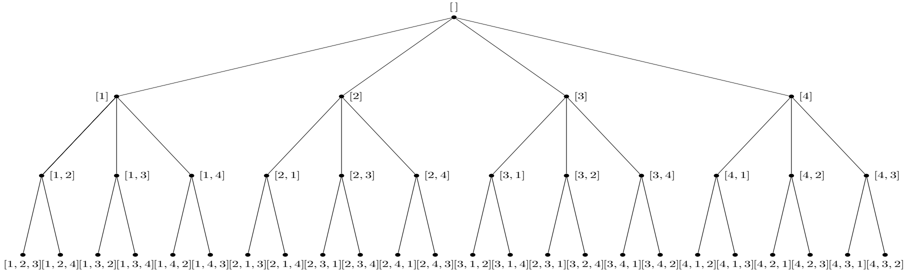

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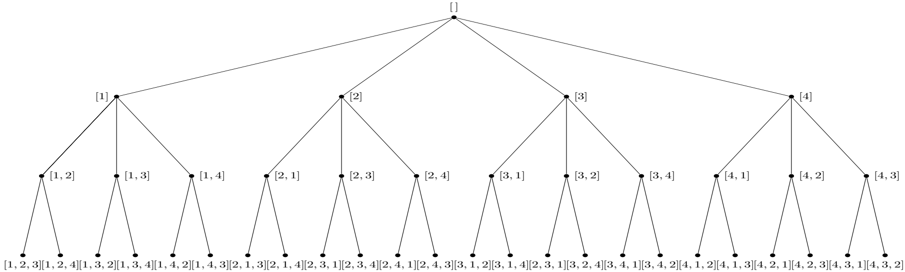

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- Motto: fail early to avoid traversing bigger subtree

Current tests

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Lemma

If $g \in N_G(H)$ *and* $[\beta_1, \beta_2, \ldots, \beta_i]^g = [\alpha_1, \alpha_2, \ldots, \alpha_i]$ $(i \leq k)$, *then* $g^{-1} H_{\beta_1, \beta_2, \ldots, \beta_i} g = H_{\alpha_1, \alpha_2, \ldots, \alpha_i}$.

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Both M and $reduced(M)$ represent H.

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Lemma

Let M be a matrix over $GF(2)$ representing H and let $g \in S_n$. Let M' be the matrix representing H^g. Then $g \in N_{S_n}(H) \iff row(M) = row(M')$.

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- Centralisers are determined identical columns - assume none

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Example (Extending from base H)

Base of $H = [1, 2, 3]$.
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Partial base image $= [3, 2, 1]$.

$$M^\sigma = \begin{array}{ccc} 3 & 2 & 1 \\ \multicolumn{3}{c}{\begin{bmatrix} 0 & 0 & 1 & & & \\ 0 & 1 & 0 & & & \\ 1 & 0 & 0 & & & \end{bmatrix}} \end{array}$$

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- If pass, we can extend the partial base image to full base image, add the corresponding group element to N; else: backtrack

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$$M = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Partial base image $= [3, 2, 1]$.

$$\begin{array}{cccccc} 3 & 2 & 1 & 5 & 4 & 6 \end{array}$$
$$M^\sigma = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Columns of M^σ must be columns of M. Extend to base image $[3, 2, 1, 5, 4]$.

- If pass, we can extend the partial base image to full base image, add the corresponding group element to N; else: backtrack
- So only check up to depth length of base of H - which is $n/4$

Tests

Tests

Lemma

If $\sigma \in N_K(H)$ *s.t.* $[\beta_1, \beta_2, \ldots, \beta_i]^\sigma = [\alpha_1, \alpha_2, \ldots, \alpha_i]$ *then* $row(M(H_{\beta_1,\beta_2,\ldots,\beta_i})^\sigma) = row(M(H_{\alpha_1,\alpha_2,\ldots,\alpha_i}))$

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Example

$$M = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Base of $H = [1, 2, 3]$.

Tests

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Base of $H = [1, 2, 3]$.Try partial base image $:= [1, 3]$.

Tests

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Base of $H = [1, 2, 3]$.Try partial base image $:= [1, 3]$.

$$M(H_{1,2}) = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

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$$M = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Base of $H = [1, 2, 3]$.Try partial base image := $[1, 3]$.

$$M(H_{1,2}) = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \qquad M(H_{1,3}) = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Tests

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Matrix has three 1's.

Tests

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Matrix has three 1's. Matrix has four 1's $\implies$ Backtrack!

Tests

Lemma

$\sigma \in K$.If L is a set of linearly dependent columns of M and $\sigma \in N_K(H)$, then L^σ is also a set of linearly dependent columns of M^σ.

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$$\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\multicolumn{6}{c}{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}} \\
c_1 & c_2 & c_3 & c_4 & c_5 & c_6
\end{array}$$

$$\begin{aligned}
c_1 + c_2 + c_4 &= [0, 0, 0] \\
c_2 + c_3 + c_5 &= [0, 0, 0] \\
c_1 + c_2 + c_3 + c_6 &= [0, 0, 0]
\end{aligned}$$

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$$\begin{array}{cccccc} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{array}$$

$$\begin{aligned} c_1^\sigma + c_2^\sigma + c_4^\sigma &= [0,0,0] \\ c_2^\sigma + c_3^\sigma + c_5^\sigma &= [0,0,0] \\ c_1^\sigma + c_2^\sigma + c_3^\sigma + c_6^\sigma &= [0,0,0] \end{aligned}$$

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$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \multicolumn{6}{c}{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}} \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{array}$$

Base of $H = [1, 2, 3]$
Try partial base image $:= [1, 3]$

$$\begin{aligned} c_1^\sigma + c_2^\sigma + c_4^\sigma &= [0, 0, 0] \\ c_2^\sigma + c_3^\sigma + c_5^\sigma &= [0, 0, 0] \\ c_1^\sigma + c_2^\sigma + c_3^\sigma + c_6^\sigma &= [0, 0, 0] \end{aligned}$$

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$$c_1^\sigma + c_2^\sigma + c_4^\sigma = [0,0,0]$$

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$$\begin{array}{cccccc} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{array}$$

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Base of $H = [1,2,3]$
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$$\begin{aligned} c_1^\sigma + c_2^\sigma + c_4^\sigma &= [0,0,0] \\ c_1 + c_3 + c_4^\sigma &= [0,0,0] \\ c_4^\sigma &= [1,0,1] \end{aligned}$$

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$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \end{array}$$

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$$\begin{aligned} c_1^\sigma + c_2^\sigma + c_4^\sigma &= [0,0,0] \\ c_2^\sigma + c_3^\sigma + c_5^\sigma &= [0,0,0] \\ c_1^\sigma + c_2^\sigma + c_3^\sigma + c_6^\sigma &= [0,0,0] \end{aligned}$$

Base of $H = [1,2,3]$
Try partial base image $:= [1,3]$

$$\begin{aligned} c_1^\sigma + c_2^\sigma + c_4^\sigma &= [0,0,0] \\ c_1 + c_3 + c_4^\sigma &= [0,0,0] \\ c_4^\sigma &= [1,0,1] \end{aligned}$$

This is not a column in M,
$\implies$ Backtrack!

Test results

Tested on 500 random groups on 20 points in GAP

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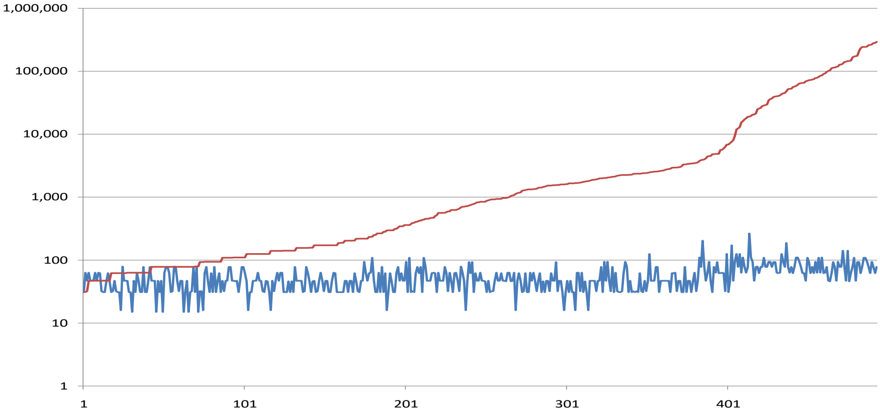
Tested on 500 random groups on 20 points in GAP



- Red: Log(time taken by the original algorithm of GAP, in milliseconds);
- Blue: Log(time taken by new algorithm)