On rational irreducible characters of finite groups

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Definition

Let χ be a complex character of G. The field generated by all $\chi(x), x \in G$ is denoted by $\mathbb{Q}(\chi)$. The character χ is called rational if $\mathbb{Q}(\chi) = \mathbb{Q}$. The group G is called a $\mathbb{Q}$-group if every irreducible complex of G is rational.

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Examples

Symmetric group $\mathbb{S}_n$, in general the Weyl group of the complex Lie algebras.

Equivalent definition

A group G is $\mathbb{Q}$-group if and only if for every $x \in G$ of order n the elements x and x^m are conjugate in G whenever $(m, n) = 1$. Equivalently for each $x \in G$ the isomorphism $\frac{N_G(<x>)}{C_G(<x>)} \cong \mathbb{A}ut(< x >)$ holds. In this case x is called a rational element.

Result 1

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If G is a solvable $\mathbb{Q}$-group, then $\pi(G) \subseteq \{2,3,5\}$ where $\pi(G)$ is the set of prime divisors of $|G|$.
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Result 3

Non-abelian composition factors of a $\mathbb{Q}$-group are isomorphic to:

$$\mathbb{A}_n;\, n \geq 5, PSP_4(3), SP_6(2), {O_8}^+(2), PSL_3(14), PSU_4(3)$$

In particular the only non-abelian simple $\mathbb{Q}$-groups are: $SP_6(2)$ and ${O_8}^+(2)$.
[W. Feit and G. M. Seitz, On finite rational groups and related topics, Illinois J. Math, 33, No. 1, 103-131 (1988).]

An old conjecture

The Sylow 2-subgroups of $\mathbb{Q}$-groups are $\mathbb{Q}$-groups. The conjecture is false. [I. M. Isaacs and G. Navaro in Sylow 2-subgroups of rational solvable groups, Math. Z. 272(2012), no. 3-4, 937-945.]

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In the groups of order $2^9 \times 3$ constructed above a Sylow 2-subgroup has nilpotency class 3. It is interesting to look at rational groups with a Sylow 2- subgroup of nilpotency class 2.

Theorem 1

Let G be a solvable rational group and $P \in Syl_2(G)$ with $cl(P) \leq 2$ and $K \in Syl_3(G)$. If G' is nilpotent, then G is a 2, 3-group and $G \cong K \rtimes P$ and we have the following:

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If G' is abelian, then $G \cong E(3^k) \rtimes P$ for some k, and G contains a normal elementary abelian 2-subgroup H such that $\frac{G}{H} \cong E(3^m) \rtimes E(2^n)$, for some $m, n \geq 0$.

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b

If G' is nonabelian and $K \in Syl_3(G)$, then K is nonabelian. moreover if $cl(P) = 2$ and $H \in Syl_2(G')$, then H is abelian.

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If G is a $\{2,5\}$-group, then $\frac{G}{O_2(G)} \cong \Pi_{i=1}^{k} M_i$, where $M_i \cong E(5^2) : \mathbb{Q}_8$.

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If G is a $\{2,5\}$-group, $K \in Syl_3(G')$, $H \in Syl_2(G')$ and $H \leq P$ then $K \trianglelefteq G'$ if and only if $P' = H$.

If G is a non-solvable rational group, $P \in Syl_2(G)$, $cl(P) \leq 2$.then every non-abelian composition factor of G is isomorphic to $\mathbb{A}_n$ for $n = 5, 6, 7$. [S. Jafari and H. Sharifi, On rational groups with Sylow 2-subgroups of nilpotency class at most two]

Definition

A Frobenius group G is a group with a subgroup H such that $1 \neq H < G$ and $H \cap H^x = 1$ for all $x \in G - H$. The subgroup H is called Frobenius complement. It is well-known that G has a normal subgroup K, Frobenius kernel, such that $G = HK$, $H \cap K = 1$.

Theorem2

If G if a Frobenius $\mathbb{Q}$-group, then exactly one of the following occurs:
(1). $G \cong E(3^n) : \mathbb{Z}_2$, $n \geq 1$, where $\mathbb{Z}_2$ acts on $E(3^n)$ by inversing each non-identity element.
(2). $G \cong E(3^{2m}) : \mathbb{Q}_8$, $m \geq 1$. where $\mathbb{Q}_8$ is the quaternion group of order 8.
(3). $G \cong E(5^2) : \mathbb{Q}_8$.
[M. R. Darafsheh and H. Sharifi, Frobenius Q-groups, Arch. Math. 83 (2004) 102-105]

Definition

A finite group G is called a 2-Frobenius group if it has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\frac{G}{H}$ and K are Frobenius groups with kernels $\frac{K}{H}$ and H respectively.

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A 2-Frobenius group is a solvable group. We proved that if G is a 2-Frobenius $\mathbb{Q}$-group, then there is a normal subgroup N of G such that $\frac{G}{N} \cong \mathbb{S}_4$.
[M. R. Darafsheh, A. Iranmanesh and S. A. Moosavi, 2-Frobenius Q-group, Indian J. pure appl. Math., 40(1), 2009]

Some generalizations

Definition

Let G be a finite group. An element $g \in G$ is called semi-rational if there exists a positive integer m such that every generator of $< x >$ is conjugate in G to either x or x^m. In the case $m = -1$, x is called inverse semi-rational. G is called semi-rational (or inverse semi-rational) if every element of G is semi-rational(or inverse semi-rational).

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Theorem 3

Let G be a finite semi-rational solvable group, then:

$$\pi(G) \subseteq \{2, 3, 5, 7, 13, 17\}$$

and if G is inverse semi-rational then $17 \notin \pi(G)$.
[D. Chillag and S. Dolfi, Semi-rational solvable groups, J. Group theory]

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if $|H|$ is odd, then $H \cong \mathbb{Z}_3$ and $|K| = 2^a \times 7^b > 1$ with $a, b \geq 0$, if $b \geq 1$ then K is not semi-rational.

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c

$|\pi(G)| \leq 5$.

[S. H. Alavi, A. Daneshkhah and M. R. Darafsheh, On semi-rational Frobenius groups, J. Alg. and its Application, Vol. 15, No. 2 (2016)]

Theorem 5

If G is a semi-rational simple group, then G has a known structure. If G is not an alternating group, then $|\pi(G)| \leq 5$ and if G is a group of Lie type, then $|\pi(G)| \leq 8$.
[S. H. Alavi and A. Daneshkhah, On semi-rational finite simple groups, Monatsh Math]

Another generalization is the following concept:

Definition

A finite group G is called a $\mathbb{Q}_1$-group if all of its non-linear characters are rational valued. obviously every $\mathbb{Q}$-group is a $\mathbb{Q}_1$-group and every abelian group is a $\mathbb{Q}_1$-group.

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Examples

$\mathbb{A}_4$, The $\mathbb{Z}\mathbb{S}$-metacyclic group of order 12.

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If G is a $\mathbb{Q}_1$-group and $N \trianglelefteq G$, then $\frac{G}{H}$ is a $\mathbb{Q}_1$-group.

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Lemma 4

If G is a $\mathbb{Q}'_1$-group, then $\exists x \in G$ such that $|\frac{G}{G'}| = |C_G(x)|$.

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Every finite group containing anticentral elements is solvable.
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Every Q_1'-group is solvable.

Theorem 7

Let G be a non-abelian Q_1'-group such that $\frac{G}{G'}$ is a cyclic p-group of odd order. Then $|C_G(x)| = |\frac{G}{G'}|$ for $x \in G - G'$.
[M. R. Darafsheh, A. Iranmanesh and S. A. Moosavi, A rational property of the irreducible characters of a finite group, LMS LN no. 387, pp. 224-227 ed. by C.M. Campbell et al.]

Definition

A non-abelian group G is called a Camina group if the conjugacy class of every element $g \in G - G'$ is gG'.

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A group satisfying the conditions of the theorem is a Camina group.

By
[A. S. Muktibodh and S. H. Ghate, On Camina group and its generalizations, Math. Bechik, 65. 2 (2013) 250-260]
One of the following possibilities holds:

(a) G is a p-group.
(b) G is a Frobenius with kernel G'.
(c) G is Frobenius with complement isomorphic to $\mathbb{Q}_8$.

Only the second possibility above holds.

Theorem 8

Let G be a non-abelian solvable Q_1'-group. Then there is a normal subgroup K of G and the following hold:

(1) $\frac{G}{K}$ is a non-abelian 2-group.

(2) $\frac{G}{K}$ is a Frobenius group with cyclic complement.

[M. R. Darafsheh, A. Iranmanesh and S. A. Moosavi, Groups whose non-linear irreducible characters are rational valued, Arch. Math.]

Theorem 9

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(1) $G \cong E(p^n) : \mathbb{Z}_t$, where p is an odd prime, $n \geq 1$ and $t \geq 1$ is even.

(2) $G \cong G' : \mathbb{Z}_t$, where G' is a rational 2-group and $t \geq 1$ is odd.

(3) $G \cong E(5^2) : \mathbb{Q}_8$ or $G \cong E(3^{2m}) : \mathbb{Q}_8$, where $m \geq 1$.

(4) $G \cong E(p^n) : H$, where p is a Fermat prime, $n \geq 1$ and H is a metacyclic group of order $2^m q$, for some Fermat prime q and $m \geq 1$.

[M. Nooz-Abadian and H. Sharifi, Frobenius Q_1-groups, Arch. Math, 105 (2015), 509-517]

$\mathbb{Q}_1$-groups with exactly one or two non-linear irreducible character.

[G. M. Seitz, Finite groups having only one irreducible representation of degree greater than one, Proc. Amer. Math. Soc. , 19 (1968), 459-461]

Theorem 10

Suppose that G has only the character of degree 1 with multiplicity m, and one irreducible character of degree n, then:
(a) G is an extra-special 2-group, $|G| = 2^{2k+1}$, $m = 2^{2k}$, $n = 2^k$.
(b) $|G| = q(q-1)$, $|G'| = q$, $m = n = q-1$, $q = p^2$, p prime.

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(b) $|G| = q(q-1)$, $|G'| = q$, $m = n = q-1$, $q = p^2$, p prime.

The groups in (a) are $\mathbb{Q}$-groups. But the group in (b) is isomorphic to the one dimensional affine group $Af_1(q) = G$, $\frac{G}{G'} \cong \mathbb{Z}_{q-1}$, hence if $q > 3$ the group $Af_1(q)$ is a $\mathbb{Q}_1'$-group.

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If the degrees of these two characters are diffrent, then:
(1) G extra-special 2-group.
(2) $G \cong Af_1(q)$, q prime power.
(3) $G \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Q}_8$.
[Y. Berkovich, D. Chillag and M. Herzog, Finite groups in which the degrees of the non-linear irreducible characters are distinct, Proc. Amer. Math. Soc. , Vol. 115 , No. 4 (1992)]

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[Y. Berkovich, D. Chillag and M. Herzog, Finite groups in which the degrees of the non-linear irreducible characters are distinct, Proc. Amer. Math. Soc. , Vol. 115 , No. 4 (1992)]

The groups in (1) and (3) are $\mathbb{Q}$-groups, and the group in (2) has only one non-linear irreducible character.

Let $[G : G'] = k$. Then the number of irreducible characters of G is k+2, which must be equal to the number of conjugacy classes of G.

$$G = g_1 G' \cup g_2 G' \cup \cdots \cup g_k G'$$

$\{g_1 = 1, g_2, \cdots, g_k\}$ a set of left transversals of G' in G. Each $g_i G'$ is a union of conjugacy classes of G. The following cases arise:

Case1. G' is the union of 3 conjugacy classes of G.

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In this case each $g_i G'$, $i \geq 2$, is a conjugacy class of G, therefore G is a Camina group. Hence G is 3-Camina group.

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Theorem 11

G is a Camina 3-group iff
(1) $G \cong \mathbb{Z}_p^r : \mathbb{Z}_{\frac{p^r-1}{2}}$ Frobenius group, p odd prime.
(2) $G \cong$ extra special 3-group.
(3) $G \cong \mathbb{Z}_3^2 : Q_8$, Frobenius group.

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Only the group in (1) is a $\mathbb{Q}_1$-group with 2 non-linear irreducible character.

Case 2. G' is the union of two conjugacy classes.

If $Z(G) = 1$, G has exactly one non-linear irreducible character.
[M. Shahriari and M. A. Shahabi, Subgroups which are the union of two conjugacy classes, Bull. Iranian Math. Soc. , Vol. 25, No. 1, 59-71 (1999)]

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Let $Z(G) \neq 1$, and G has 2 non-linear irreducible characters of degree n. If $1 \neq z \in Z(G)$ and χ is a non-linear irreducible character, then

$$\chi|_{Z(G)(z)} = \lambda(z)\chi(1)$$

where λ is an irreducible character of $Z(G)$. But $Z(G)$ is elementary abelian 2-group, therefore

$$\lambda(z) = \pm 1 \text{ and } \chi|_{Z(G)} = \pm\chi(1) = \pm n$$

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Orthogonality of columns: $na + nb = 0 \Rightarrow b = -a$
Orthogonality of rows: $n^2 \cdots - n^2 - a^2 - \cdots = 0$
Therefore $|Z(G)| = 2$ and $a = \cdots = 0$. Each irreducible character outside $Z(G)$ is zero.

Definition

A finite group G is called a $\mathbb{VZ}$-group if all non-linear irreducible characters of G vanish off the center of G.
[M. L. Lewis, character tables of groups where all non-linear irreducible characters vanish off the center, Ischia Group Theory 2008, Edited by M. Bianchi, P. Longobardi, M. Maj and C. M. Scoppola, World Scientific, 2009, 174-182.]

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Let G be a $\mathbb{VZ}$-group, then $G' \leq Z(G)$, G is nilpotent of class 2. Therefore G is a 2-group with $G' = Z(G) \cong \mathbb{Z}_2$, implying G is extra-special, a contradiction.

Let G be a $\mathbb{Q}_1$-group with exactly two non-linear irreducible characters. Then:

(1) $G \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) : \mathbb{Q}_8$.

(2) $G \cong \mathbb{Z}_p^r : \mathbb{Z}_{\frac{p^r-1}{2}}$ Frobenius group, p odd prime.

Thank You!