Beauville groups: bigger than you think!

Ben Fairbairn
Birkbeck, University of London

Groups St Andrews, 6^{th} August 2017

I'm an idiot because Beauville groups

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Characteristically Simple Groups

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A finite group G is **characteristically simple** if there exists a simple group S such that $G = S^k$ for some positive integer k.

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- Any simple group
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- $E_8(27)^{1320}$, $L_{101}(101^{101})^{101}$, $HS^{244823040}, \ldots$

2-generated Characteristically Simple Groups

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We say that two generating pairs $(x, y), (x', y') \in S^2$ are equivalent of there is an automorphism $\alpha \in \mathrm{Aut}\ S$ such that $x^\alpha = x'$ and $y^\alpha = y'$. Hall showed that if S is simple, then $h(S)$ is the number of equivalence classes under this relation.

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Theorem (Morgan, Roney-Dougal '15)

For $n \geq 14$

$$\left(1 - \frac{1}{n} - \frac{7 \cdot 5}{n^2}\right)\left(\frac{n!}{4}\right) < h(A_n) < \left(1 - \frac{1}{n} - \frac{0 \cdot 93}{n^2}\right)\left(\frac{n!}{4}\right)$$

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Theorem (F. '17+)

If S is a simple group and $k \leq h(S)$, then there exists a generating pair of S^k, $x, y \in S^k$ such that there exists an automorphism $\phi \in Aut\ S^k$ such that

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Smaller k? We can come down an even number of steps by neglecting non-simply invertible pairs (or pairs of simply invertible pairs). We can come down an odd number of steps if there exists at least one simply invertible pair. It can be shown that such pairs always exist. $\square$

Beauville I: Definition

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A **Beauville structure** for the group G is a set of pairs of elements $\{\{g_1, h_1\}, \{g_2, h_2\}\} \subset G \times G$ with the property that $\langle g_1, h_1 \rangle = \langle g_2, h_2 \rangle = G$ such that

$$\Sigma(g_1, h_1) \cap \Sigma(g_2, h_2) = \{e\}.$$

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Theorem (Garion, Larson, Lubotzky; Guralnick, Malle; F., Magaard & Parker, '10-'13)

Every non-abelian finite simple group apart from A_5 *is a Beauville group.*

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Theorem (Garion, Larson, Lubotzky; Guralnick, Malle; F., Magaard & Parker, '10-'13)

Every non-abelian finite simple group apart from A_5 is a Beauville group.

Theorem (Jones '15)

Let S be a simple group that is either an alternating group; $L_2(q)$; $L_3(q)$; $U_3(q)$; ${}^2B_2(2^{2n+1})$; ${}^2G_2(3^{2n+1})$ or a sporadic group. Then S^k is a Beavuille group if and only if it is 2-generated and $(S,k) \neq (A_5, 1)$.

Beauville III: Motivation

- Beauville structures give us 'Beauville surfaces' which have really nice properties like...
- Easy to work with fundamental groups and automorphism groups; rigidity; they're of general type etc.
- Consequently, they're useful e.g....
  - Cheap counterexamples to the Friedman-Morgan conjecture (don't ask).
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(There are speculative links to the theory of buildings thanks to the work of Cartwright and Steger (and Howie?).)

Beauville IV: A Further Definition

Definition (Bauer, Catanese, Grunewald '04)

A Beauville structure $\{\{g_1, h_1\}, \{g_2, h_2\}\} \subset G \times G$ and its corresponding Beauville group G are **strongly real** if there exist automorphisms $\phi_1, \phi_2 \in AutG$ and elements $z_1, z_2 \in G$ such that for $i = 1, 2$

$$\begin{aligned} z_i \phi_i(g_i) z_i^{-1} &= g_i^{-1} \text{ and} \\ z_i \phi_i(h_i) z_i^{-1} &= h_i^{-1}. \end{aligned}$$

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(Basically the Beauville surface has even nicer properties if the structure has this extra property. There are also links to symmetric Riemann surfaces and Klein surfaces.)

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$$r_p(T) := \{r_p(X), r_p(Y), r_p(XY)\}.$$

Lemmas

Lemma (easy)

Let $X_i = (x_{i1}, x_{i2}, \ldots, x_{ik})$ *and* $Y_i = (y_{i1}, y_{i2}, \ldots, y_{ik})$ *for* $i = 1, 2$ *be generating pairs for* S^k

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Example: $L_2(q)^k$ where $(k, q) \neq (1, 4)$, $(1, 5)$

Theorem

The characteristically simple group $L_2(q)^k$ is a strongly Beauville group if and only if $L_2(q)^k$ is 2-generated and $(k, q) \neq (1, 4)$, $(1, 5)$.

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Idiot I: $H \times H$

Strongly Real Beauville Groups

Ben Fairbairn

simple Beaville groups are in general strongly real.

Question 1 *Which characteristically simple Beauville groups are strongly real?*

As a more specific conjecture on these matters we assert the following.

Conjecture 3 *If H is a finite simple group of order greater than 3, then the group $H \times H$ is a strongly real Beauville group.*

Note that Corollary 8 tells us that this is true for all abelian characteristically simple

$_{23}$. Similarly $A_5 \times A_5 \times A_5$ is not a strongly real Beauville group.

ension of Conjecture 3 to products of a larger number of copies

Idiot II: The Moron paper

More on Strongly Real Beauville Groups

Ben Fairbairn

Table 1 Values of k such that every a simple group H with $|H| < 100{,}000$ the group H^r is a strongly real Beauville group for every $r \leq k$

H	k	H	k	H	k	H	k	H	k	H	k
A_5	0	$L_2(7)$	2	A_6	2	$L_2(8)$	4	$L_2(11)$	4	$L_2(13)$	4
$L_2(17)$	6	A_7	14	$L_2(19)$	6	$L_2(16)$	2	$L_3(3)$	14	$U_3(3)$	6
$L_2(23)$	2	$L_2(25)$	10	M_{11}	0	$L_2(27)$	12	$L_2(29)$	12	$L_2(31)$	14
A_8	18	$L_3(4)$	4	$U_4(2)$	6	${}^2B_2(8)$	52	$L_2(32)$	2	$L_2(41)$	18
$L_2(43)$	18	$L_2(47)$	22	$L_2(49)$	22	$U_3(4)$	28	$L_2(53)$	48	M_{12}	16

Idiot III: $h(A_7)$

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A_7^{916}

$$
\begin{aligned}
X_1 &:= (1,2,3,4,5,6,7)(8,14,11)(9,13)(10,12)(15,16,17,18,19,20,21)\\
&\quad (22,23,27,28)(24,26)(30,34,33,32,31)(36,42,37,39,41)\\
Y_1 &:= (1,4,7)(2,6)(3,5)(8,14,13,12,11,10,9)(15,19,17,21,16,18,20)\\
&\quad (23,24,25,26,27)(29,35,34,30)(31,33)(37,38,40,41,39)\\
X_2 &:= (1,2,6,7)(3,5)(9,13,12,11,10)(15,21,16,18,20)\\
&\quad (22,23,24,25,26,27,28)(29,35,32)(30,34)(31,33)(36,37,38,39,40,41,42)\\
Y_2 &:= (2,3,4,5,6)(8,14,13,9)(10,12)(16,17,19,20,18)\\
&\quad (22,25,28)(23,27)(24,26)(29,35,34,33,32,31,30)(36,40,38,42,37,39,41)\\
\phi &:= (1,7)(2,6)(3,5)(8,14)(9,13)(10,12)(15,21)(16,20)\\
&\quad (17,19)(22,28)(23,27)(24,26)(29,35)(30,34)(31,33)(36,42)(37,41)(38,40)\\
z_1 &:= e\\
z_2 &:= e
\end{aligned}
$$

A further simply invertible generating pair for A_7 is given by (1,2,3,4,5,6,7), (2,3,4,5,6) which is inverted by conjugation by (17)(26)(35).

Some conjectures

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Bauer, Catanese, Grunewald '05)

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A finite non-abelian characteristically simple group G is a strongly real Beauville group if and only if G is 2-generated and not one of A_5, M_{11} or M_{23}.

Conclusion

Figure: An idiot.

Thanks for Listening!