Partial linear spaces with a primitive rank 3 automorphism group of affine type

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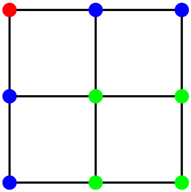

Joint work with John Bamberg, Alice Devillers, and Cheryl Praeger

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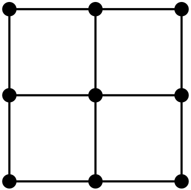

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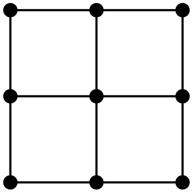

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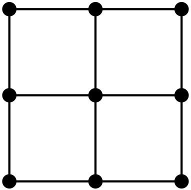

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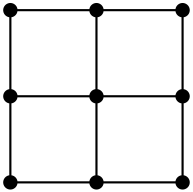

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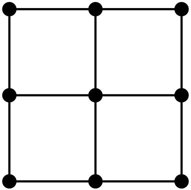

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A partial linear space is proper if it is not a linear space or a graph.

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Such partial linear spaces are:

- point-transitive, line-transitive, flag-transitive

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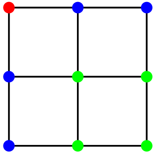

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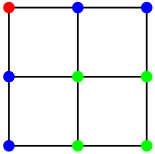

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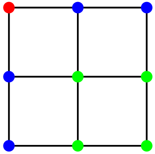

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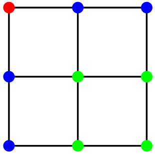

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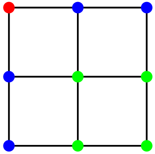

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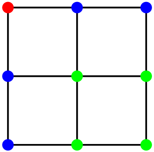

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N.B. A graph with an imprimitive rank 3 group of automorphisms is either $K_{n[m]}$ or $n \cdot K_m$. No such result for a proper PLS!

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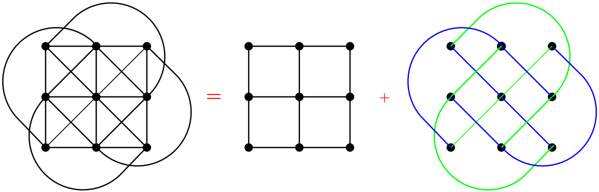
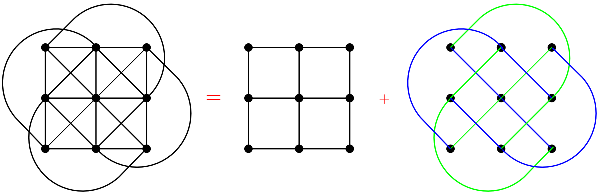
- Focus on the affine case.

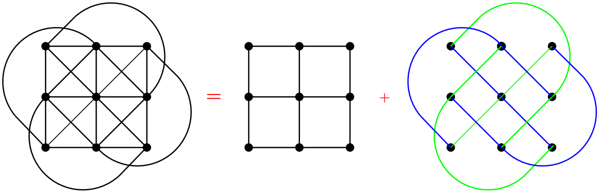




Let G be a rank 3 permutation group on a set P. TFAE:

(i) There is a linear space S on P such that $G \leqslant \mathrm{Aut}(S)$ and G has two orbits on lines.

(ii) There are 2 partial linear spaces S_1 and S_2 on P with different collinearity relations such that $G \leqslant \mathrm{Aut}(S_i)$.



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- Biliotti-Montinaro-Francot (2015): classified 2-$(v, k, 1)$ designs with a primitive rank 3 affine group on points that has two orbits on lines (except for certain groups).

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Theorem (Bamberg, Devillers, F., Praeger)

Let S be a proper partial linear space and $G \leqslant \mathrm{Aut}(S)$ a rank 3 primitive permutation group with socle $V = V_d(p)$. Then

(i) *S lies in one of the 4 infinite families just discussed, or*

(ii) *S lies in a (known) finite list, or*

(iii) *one of the following holds:*

  (a) $G_0 \leqslant \Gamma\mathrm{L}_1(p^d)$, *or*
  (b) $V = V_n(p) \oplus V_n(p)$ *and* $G_0 \leqslant \Gamma\mathrm{L}_1(p^n) \wr C_2$ *where* $d = 2n$, *or*
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Open problem

Classify the rank 3 proper PLSs in (iii). Especially (c).

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1. $X \cup \{0\}$ is not a subspace of $V_d(p)$.
2. If $x, y \in \ell \setminus \{0\}$ and $x \neq y$, then $y - x \in X$.

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- If $\ell \subseteq \langle x \rangle_{\mathbb{F}_q}$, then Kantor's classification of 2-transitive linear spaces $\implies$ $\ell = \langle x \rangle_{\mathbb{F}_r}$ for some subfield $\mathbb{F}_r$ of $\mathbb{F}_q$, and Example (2) holds.

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- Otherwise, there exists $y \in \ell \setminus \langle x \rangle_{\mathbb{F}_q}$. Now $y^{G_{0,x}} \subseteq \ell$.
- Repeat this process until you find examples or some $u \neq v \in \ell \setminus \{0\}$ such that $u - v \notin x^{G_0}$ a contradiction.

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Let $V = V_d(3)$. Suppose that either

1. $d = 5$ and $G_0 = M_{11}$ with subdegrees 132 and 110, or
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Then there is a partial linear space S with $\mathrm{Aut}(S) = V : G_0$.

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Then there is a partial linear space S with $\mathrm{Aut}(S) = V : G_0$.

- Each point lies on 12 lines.
- The former has $243 = |V|$ lines and line size 12.
- The latter has $162 = 2|V|$ lines and line size 6.