2-arc-transitive digraphs

Michael Giudici

Centre for the Mathematics of Symmetry and Computation

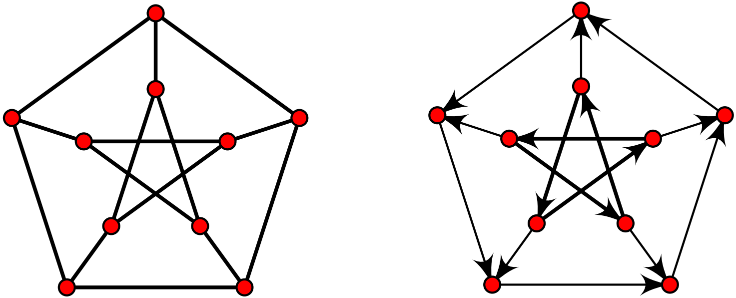
THE UNIVERSITY OF WESTERN AUSTRALIA

Groups St Andrews

Birmingham, August 2017

on joint work with Cai Heng Li and Binzhou Xia

Graphs and digraphs



A graph is a symmetric non-reflexive relation A on a set V.
Write $u \sim v$.

A digraph is an asymetric non-reflexive relation A on a set V.
Write $u \to v$.

Automorphism groups

V is the vertex set, A is the arc set

$\mathrm{Aut}(\Gamma)$ is the set of all permutations in $\mathrm{Sym}(V)$ that fixes A setwise.

vertex-transitive, arc-transitive

s-arcs

An s-arc in a graph is $v_0 \sim v_1 \sim v_2 \sim \cdots \sim v_s$ with $v_i \neq v_{i+1}$.

s-arcs

An s-arc in a graph is $v_0 \sim v_1 \sim v_2 \sim \cdots \sim v_s$ with $v_i \neq v_{i+1}$.

An s-arc in a digraph is $v_0 \to v_1 \to v_2 \cdots \to v_s$.

s-arcs

An s-arc in a graph is $v_0 \sim v_1 \sim v_2 \sim \cdots \sim v_s$ with $v_i \neq v_{i+1}$.

An s-arc in a digraph is $v_0 \to v_1 \to v_2 \cdots \to v_s$.

Say Γ is (G, s)-arc-transitive if G is transitive on the set of s-arcs.

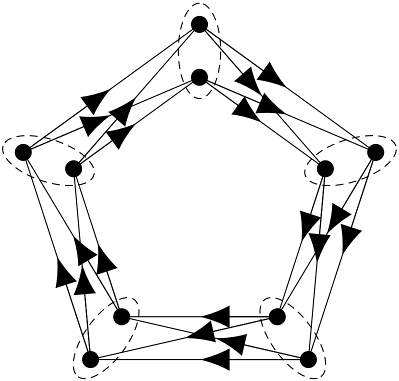
Bounding s

- Cycles and directed cycles are s-arc-transitive for all s.
- Weiss (1981): A graph of valency at least 3 is at most 7-arc-transitive.

Bounding s

- Cycles and directed cycles are s-arc-transitive for all s.
- Weiss (1981): A graph of valency at least 3 is at most 7-arc-transitive.
- Praeger (1989): For all $k, s \geq 2$ there are infinitely many s-arc-transitive digraphs that are not $(s+1)$-arc-transitive.

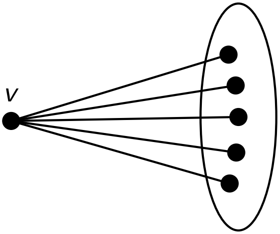
Example



Local actions-graphs

Let Γ be G-arc-transiive graph.

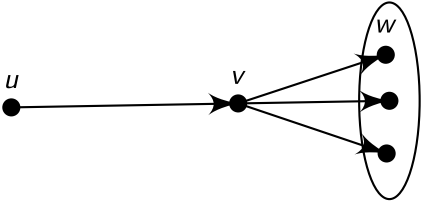
Then it is $(G, 2)$-arc-transitive if and only if $G_v^{\Gamma(v)}$ is 2-transitive.



Local actions-digraphs

Let Γ be a G-arc-transitive digraph.

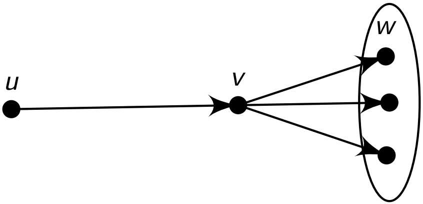
Then it is $(G, 2)$-arc-transitive if and only if $G_v = G_{uv} G_{vw}$.



Local actions-digraphs

Let Γ be a G-arc-transitive digraph.

Then it is $(G, 2)$-arc-transitive if and only if $G_v = G_{uv} G_{vw}$.



Will then be 3-arc-transitive if and only if $G_{uv} = G_{xuv} G_{uvw}$.

Products of digraphs

Let Γ be a digraph with vertex set V.

Then Γ^n is the digraph with vertex set V^k and

$$(u_1, \ldots, u_n) \to (v_1, \ldots, v_n)$$

if and only if $u_i \to v_i$ for all i.

Products of digraphs

Let Γ be a digraph with vertex set V.

Then Γ^n is the digraph with vertex set V^k and

$$(u_1, \dots, u_n) \to (v_1, \dots, v_n)$$

if and only if $u_i \to v_i$ for all i.

Lemma If Γ is (G, s)-arc-transitive then Γ^n is $(G \wr S_n, s)$-arc-transitive.

Existence Question

Question (Praeger 1989): Does there exist a vertex-primitive 2-arc-transitive digraph?

Coset digraphs

G a group, $H \leqslant G$, $g \in G$ such that $g^{-1} \notin HgH$.

$\Gamma = \mathrm{Cos}(G, H, HgH)$ is the digraph defined by

- vertices are right cosets of H.
- $Hx \to Hy$ if $yx^{-1} \in HgH$.

G acts on Γ by right multiplication as a group of automorphisms

Γ is connected if and only if $\langle H, g \rangle = G$.

An Example

Giudici-Li-Xia (2017)

- $G = \mathrm{PSL}(3, p^2)$ for $p \equiv \pm 2 \pmod 5$, with $p \neq 3$.
- $H \cong A_6$, a maximal subgroup
- H has two conjugacy classes of A_5's. Take K_1, K_2 from different conjugate classes.
- There exists $g \in G$ such that $K_1^g = K_2$ and $g^{-1} \notin HgH$.
- Let $\Gamma = Cos(G, H, HgH)$

An Example

Giudici-Li-Xia (2017)

- $G = \mathrm{PSL}(3, p^2)$ for $p \equiv \pm 2 \pmod 5$, with $p \neq 3$.
- $H \cong A_6$, a maximal subgroup
- H has two conjugacy classes of A_5's. Take K_1, K_2 from different conjugate classes.
- There exists $g \in G$ such that $K_1^g = K_2$ and $g^{-1} \notin HgH$.
- Let $\Gamma = Cos(G, H, HgH)$

Γ is a $(G, 2)$-arc transitive vertex-primitive digraph

An Example

Giudici-Li-Xia (2017)

- $G = \mathrm{PSL}(3, p^2)$ for $p \equiv \pm 2 \pmod 5$, with $p \neq 3$.
- $H \cong A_6$, a maximal subgroup
- H has two conjugacy classes of A_5's. Take K_1, K_2 from different conjugate classes.
- There exists $g \in G$ such that $K_1^g = K_2$ and $g^{-1} \notin HgH$.
- Let $\Gamma = Cos(G, H, HgH)$

Γ is a $(G, 2)$-arc transitive vertex-primitive digraph

Not 3-arc-transitive.

An Example

Giudici-Li-Xia (2017)

- $G = \mathrm{PSL}(3, p^2)$ for $p \equiv \pm 2 \pmod 5$, with $p \neq 3$.
- $H \cong A_6$, a maximal subgroup
- H has two conjugacy classes of A_5's. Take K_1, K_2 from different conjugate classes.
- There exists $g \in G$ such that $K_1^g = K_2$ and $g^{-1} \notin HgH$.
- Let $\Gamma = Cos(G, H, HgH)$

Γ is a $(G, 2)$-arc transitive vertex-primitive digraph

Not 3-arc-transitive.

Also Γ^n is $(G \wr S_n, 2)$-arc-transitive and vertex-primitive.

Diagonal groups

Giudici-Xia (2018)

- T a finite nonabelian simple group, $|T| = k$
- $g = (t_1, t_2, \ldots, t_k)$ with all entries distinct
- $D = \{(t, \ldots, t) \mid t \in T\}$
- $\Gamma(T) = Cos(T^k, D, DgD)$

$\Gamma(T)$ is a $(G, 2)$-arc-transitive vertex-primitive digraph with $G = T^k \rtimes (T \rtimes \mathrm{Aut}(T))$.

Diagonal groups

Giudici-Xia (2018)

- T a finite nonabelian simple group, $|T| = k$
- $g = (t_1, t_2, \dots, t_k)$ with all entries distinct
- $D = \{(t, \dots, t) \mid t \in T\}$
- $\Gamma(T) = Cos(T^k, D, DgD)$

$\Gamma(T)$ is a $(G, 2)$-arc-transitive vertex-primitive digraph with $G = T^k \rtimes (T \rtimes \mathrm{Aut}(T))$.

Not 3-arc-transitive.

Also $\Gamma(T)^n$ is $(G \wr S_n, 2)$-arc-transitive and vertex-primitive.

Characterisation

Giudici-Xia (2018)

Theorem Let Γ be a finite (G, s)-arc-transitive vertex-primitive digraph. Then one of the following holds:

- $\Gamma \cong \Gamma(T)^n$ for some $n \geqslant 1$.
- $\Gamma \cong \Sigma^n$ for some $n \geqslant 1$ and Σ is a (H, s)-arc-transitive vertex-primitive digraph with H an almost simple group.

Characterisation

Giudici-Xia (2018)

Theorem Let Γ be a finite (G, s)-arc-transitive vertex-primitive digraph. Then one of the following holds:

- $\Gamma \cong \Gamma(T)^n$ for some $n \geqslant 1$.
- $\Gamma \cong \Sigma^n$ for some $n \geqslant 1$ and Σ is a (H, s)-arc-transitive vertex-primitive digraph with H an almost simple group.

Question: What is the largest value of s for a (G, s)-arc-transitive vertex-primitive digraph?