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Sums of element orders in finite groups

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Introduction

In this talk, G denotes a finite group, n denotes a positive integer and C_n denotes the cyclic group of order n. In 2009, H. Amiri, S.M. Jafarian Amiri and I.M. Isaacs introduced in their paper [1] the function

$$\psi(G) = \sum\{o(x) \mid x \in G\},$$

where $o(x)$ denotes the order of x. Thus $\psi(G)$ denotes the sum of element orders of the finite group G.

They proved the following basic theorem:

Theorem 1

If G is a non-cyclic group of order n, then $\psi(G) < \psi(C_n)$.

Thus C_n is the unique group of order n which attains the maximal value of $\psi(G)$ for groups of that order.

In our paper (see [3], by M.Herzog, P.Longobardi and M.Maj) we continue the study of the function $\psi(G)$. Our main results are the following two theorems, which improve the result of Theorem 1.

Theorem 2

If G is a non-cyclic finite group of order n, then

$$\psi(G) \;\leq\; \frac{7}{11}\psi(C_n).$$

Theorem 3

If G is a non-cyclic group of order n and q is the smallest prime divisor of n, then:

$$\psi(G) < \frac{1}{q-1}\psi(C_n).$$

The result of Theorem 2: $\psi(G) \leq \frac{7}{11}\psi(C_n)$ for non-cyclic groups G of order n is the best possible. Moreover, the equality $\psi(G) = \frac{7}{11}\psi(C_n)$ occurs for an infinite number of values of n.

Indeed, for each $n = 4k$, where k denotes an odd integer, there exists a group of order n satisfying $\psi(G) = \frac{7}{11}\psi(C_n)$. This follows from the next proposition.

Proposition 4

Let k be an odd integer and let $n = 4k$. Then $G = C_{2k} \times C_2$ is non-cyclic, $|C_{2k} \times C_2| = n$ and

$$\psi(C_{2k} \times C_2) = \frac{7}{11}\psi(C_n).$$

The proof of Proposition 4 is simple. It is easy to see that if H and K are subgroups of G of coprime orders, then $\psi(H \times K) = \psi(H)\psi(K)$.

Moreover,

$$\psi(C_4) = 2 \cdot 4 + 2 + 1 = 11, \ \psi(C_2 \times C_2) = 3 \cdot 2 + 1 = 7.$$

Since $\psi(C_n) = \psi(C_{4k}) = \psi(C_4 \times C_k)$, it follows that

$$\psi(C_n) = \psi(C_4)\psi(C_k), \text{ so } \psi(C_k) = \frac{1}{11}\psi(C_n).$$

Now $\psi(C_{2k} \times C_2) = \psi((C_k) \times (C_2 \times C_2))$, so

$$\psi(C_{2k} \times C_2) = \psi(C_k)\psi(C_2 \times C_2) = \frac{7}{11}\psi(C_n),$$

as required. Notice that in particular

$$\psi(C_2 \times C_2) = 7 = \frac{7}{11}11 = \frac{7}{11}\psi(C_4)$$

in agreement with Proposition 4.

However, not for all n there exists a non-cyclic groups G of order n satisfying $\psi(G) = \frac{7}{11}\psi(C_n)$.
For example, there exist only two groups of order 6: C_6 and S_3, the symmetric group of degree 3. Now $\psi(C_6) = 2 \cdot 6 + 2 \cdot 3 + 2 + 1 = 21$ and $\psi(S_3) = 2 \cdot 3 + 3 \cdot 2 + 1 = 13$, so

$$\psi(S_3) = \frac{13}{21}\psi(C_6) \ < \ \frac{7}{11}\psi(C_6),$$

since $143 < 147$.
We also mention the following important result: if p denotes a prime and $P = C_{p^n}$, then

$$\psi(P) = \frac{p^{2n+1}+1}{p+1} = \frac{p|P|^2+1}{p+1} > \frac{p}{p+1}|P|^2.$$

We now focus our attention on the second major result mentioned above:

Theorem 3

Let G be a non-cyclic finite group of order n and let q be the smallest prime divisor of n. Then: $\psi(G) < \frac{1}{q-1}\psi(C_n)$.

If n is even, then $q = 2$ and Theorem 3 implies that $\psi(G) < \psi(C_n)$, which is weaker than the claim of Theorem 2. But if n is odd, then $q \geq 3$ and Theorem 3 yields the following important corollary.

Corollary 5

Let G be a non-cyclic finite group of odd order n. Then

$$\psi(G) < \frac{1}{2}\psi(C_n).$$

Notice that for groups of odd order Corollary 5 is stronger than Theorem 2, which only claims that $\psi(G) \leq \frac{7}{11}\psi(C_n)$.

Recently S.M. Jafarian Amiri and M. Amiri in [4] and R. Shen, G. Chen and C. Wu in [7] studied **non-cyclic** finite groups G of order n with the largest value of $\psi(G)$, and obtained information about the structure of such groups G in certain cases.
Moreover, **products** of element orders of a finite group G, and some other functions on the orders of elements of G, have been recently studied by M. Garonzi and M. Patassini in [2]. In particular, they proved that if

$$\mathcal{P}(G) = \prod \{o(x) \mid x \in G\}$$

denotes the product of the orders of elements of G and $|G| = n$, then $\mathcal{P}(G) \leq \mathcal{P}(C_n)$, with equality if and only if $G = C_n$. Thus the cyclic group C_n attains the maximal value of both $\psi(G)$ and $\mathcal{P}(G)$ among groups of order n.

The minimal value of $\psi(G)$

By the previous results, the **maximal** value of $\psi(G)$ for groups of order n is attained by the unique group C_n.

The **minimal** value of $\psi(G)$ for groups of order n had been also investigated. For p-groups of order p^n, the minimal value of $\psi(G)$ is attained by groups of exponent p. Thus for 2-groups of order 2^n, the group attaining the minimal value is unique, the elementary abelian group of order 2^n. But for $p > 2$, groups of order $p^r > p^2$ attaining the minimal value are not unique. In particular, there are two groups of order 27 of exponent 3, with $\psi(G) = 79$ for both of them. . In the general case of groups of an arbitrary order n, the problem of the minimal value of $\psi(G)$ is still open.

The minimal value of $\psi(G)$ for groups of order n was also investigated in relation to the **non-abelian simple groups**. For the six simple groups of the smallest order:

$$|A_5| = 60 \quad |L_3(2)| = 168 \quad |A_6| = 360 \quad |L_2(8)| = 504$$

$$|L_2(11)| = 660 \quad |L_2(13)| = 1092$$

the respective values of $\psi(G)$ are:

$$211,\ 715,\ 1411,\ 3319,\ 3741,\ 7281.$$

For each of these values of n, the $\psi(G)$ of the simple group of order n is the **unique minimum** of the values of ψ for groups of order n. For example, if we consider groups of order 60, then $\psi(A_5) = 211$, while for other groups G of order 60 we have $\psi(G) \geq 337$, with the maximum $\psi(C_{60}) = 1617$.

These observations raised several questions. Let S denote a simple group.

Question 1.

If $|S| = n$, is $\psi(S)$ the minimal value of $\psi(G)$ for groups of order n?

The answer is "NO". For example, there are two simple groups of order 20160: A_8 and $L_3(4)$. Now $\psi(A_8) = 137047$, while $\psi(L_3(4)) = 103111$. Hence $\psi(A_8)$ is not minimal.
I don't know if $\psi(L_3(4))$ is minimal for groups of order 20160.

Question 2.

If G is a non-simple group satisfying $|G| = |S|$, does it follow that $\psi(S) < \psi(G)$?

The answer is "NO". It was shown by Y.Marefat, A.Iranmanesh and A.Tehranian in 2013 (see [6]) that if $S = L_2(64)$ and $G = 3^2 \times Sz(8)$, then $|G| = |S| = 262,080$ and $\psi(G) \leq \psi(S)$.

We conclude these remarks with our conjecture:

Conjecture

If G is a solvable group satisfying $|G| = |S|$, then $\psi(S) < \psi(G)$.

Proof of Theorem 3

We shall describe now the proof of Theorem 3. This proof is simpler than that of Theorem 2, but it includes the main ingredients of the proof of Theorem 2.

The following two propositions play an important role in our proofs. The first one is the following Corollary B of the paper [1] of H. Amiri, S.M. Jafarian Amiri and I.M. Isaacs from 2009.

Proposition 6

Let G be a group of order n and let p be the largest prime divisor of n. Suppose that P is a **cyclic normal** *Sylow p-subgroup of G. Then*

$$\psi(G) \leq \psi(P)\psi(G/P),$$

with equality if and only if P is central in G.

The second proposition describes our lower bound for the Euler's function $\varphi(n)$.

Proposition 7

Let n be an integer greater than 1, with the largest prime divisor p and the smallest prime divisor q. Then

$$\varphi(n) \geq \frac{q-1}{p} n.$$

Notice that for $n = p^r$ we get equality:

$$\varphi(n) = (p-1)p^{r-1} = \frac{p-1}{p} p^r = \frac{q-1}{p} n.$$

We return now to the proof of Theorem 3, which is now restated.

Theorem 3

Let G be a non-cyclic finite group of order n and let q be the smallest prime divisor of n. Then $\psi(G) < \frac{1}{q-1}\psi(C_n)$.

Proof We need to prove that if G is a **non-cyclic** group of order n, then $\psi(G) < \frac{1}{q-1}\psi(C_n)$.
In other words, we need to prove that if G is a group of order n satisfying

$$\psi(G) \geq \frac{1}{q-1}\psi(C_n),$$

then $G = C_n$.

Recall that $\varphi(n)$ is the number of elements of C_n of order n. Hence $\psi(C_n) > \varphi(n)n$ and by Proposition 7 $\varphi(n) \geq (q-1)n/p$, where p denotes the largest prime divisor of n. Thus by our assumptions

$$\psi(G) \geq \frac{1}{q-1}\psi(C_n) > \frac{\varphi(n)n}{q-1} \geq \frac{n^2}{p}.$$

Hence there exists $x \in G$ of order $o(x) > \frac{n}{p}$. Thus $[G : \langle x \rangle] < p$, so $\langle x \rangle$ contains a **cyclic** Sylow p-subgroup P of G.
Since P is normal in $\langle x \rangle$, it follows that $N_G(P) \geq \langle x \rangle$, which implies that $[G : N_G(P)] < p$. Hence $N_G(P) = G$ and P is a **cyclic normal** Sylow p-subgroup of G.

Thus the assumptions of Proposition 7 (P is normal and cyclic) are satisfied and it follows that

$$\psi(G) \leq \psi(P)\psi(G/P).$$

On the other hand, since $n = |G| = |P|\frac{n}{|P|}$ and $(|P|, \frac{n}{|P|}) = 1$, it follows from our assumptions that

$$\psi(G) \geq \frac{1}{q-1}\psi(C_n) = \frac{1}{q-1}\psi(C_{|P|})\psi(C_{\frac{n}{|P|}}).$$

The two inequalities concerning $\psi(G)$ imply that

$$\psi(P)\psi(G/P) \geq \psi(G) \geq \frac{1}{q-1}\psi(C_{|P|})\psi(C_{\frac{n}{|P|}}),$$

and cancellation by $\psi(P) = \psi(C_{|P|})$ yields

$$\psi(G/P) \geq \frac{1}{q-1}\psi(C_{\frac{n}{|P|}}).$$

If $n = p^r$, p a prime, then the existence of $x \in G$ satisfying $o(x) > n/p$ implies that $o(x) = n$ and G is cyclic, as required. So we may assume that $n = |G|$ is divisible by exactly k different primes with $k > 1$. Applying induction with respect to k, we may assume that the theorem holds for groups of order which has less than k distinct prime divisors.
Now $|G/P|$ has $k-1$ distinct prime divisors,
$|G/P| = \frac{n}{|P|}$ and as shown above, G/P satisfies $\psi(G/P) \geq \frac{1}{q-1}\psi(C_{\frac{n}{|P|}})$, which is our assumption. It follows by our inductive hypothesis that also G/P is cyclic. Denoting by F the cyclic complement of P in G, we deduce that

$$G = P \rtimes F.$$

Since P and F are cyclic, $|P||F| = |G| = n$ and $(|P|, |F|) = 1$, it follows that

$$\psi(C_n) = \psi(P)\psi(F).$$

If $C_F(P) = F$, then $G = P \times F$ and G is cyclic, as required.

So it suffices to prove that if $C_F(P) = Z < F$, then $\psi(G) < (1/(q-1))\psi(C_n)$, contrary to our assumptions.
We showed that under this configuration the following equality holds:

$$\psi(G) = \psi(P)\psi(Z) + |P|\psi(F \setminus Z).$$

We now replace $\psi(F \setminus Z)$ by $\psi(F)$ and get

$$\psi(G) < \psi(P)\psi(F)\left(\frac{\psi(Z)}{\psi(F)} + \frac{|P|}{\psi(P)}\right).$$

Since $\psi(C_n) = \psi(P)\psi(F)$, it follows that

$$\psi(G) < \psi(C_n)\left(\frac{\psi(Z)}{\psi(F)} + \frac{|P|}{\psi(P)}\right).$$

We concluded the proof by showing that

$$\frac{\psi(Z)}{\psi(F)} + \frac{|P|}{\psi(P)} < \frac{1}{q-1},$$

which implies that $\psi(G) < \psi(C_n)\frac{1}{q-1}$, a final contradiction. ■

Our solvability results

Let G be a non-cyclic finite group of order n and let q and p be the smallest and the largest prime divisor of n, respectively. Recall that by Theorem 3: $\psi(G) < \frac{1}{q-1}\psi(C_n)$.

But what can we say about groups of order n satisfying: $\psi(G) \geq \frac{1}{q}\psi(C_n)$? There exist non-cyclic groups satisfying this condition. For example,

$$\psi(S_3) = 13 > \frac{1}{2}\psi(C_6) = \frac{21}{2}.$$

We tackled a more general problem: which groups satisfy

$$\psi(G) \geq \frac{1}{2(q-1)}\psi(C_n).$$

Our result was:

Theorem 8

If G of order n satisfies

$$\psi(G) \geq \frac{1}{2(q-1)}\psi(C_n),$$

then it is solvable and either its Sylow p-subgroups or its Sylow q-subgroups are cyclic.

Theorem 8 implies the following corollary:

Corollary 9

If $|G| = n$ and either $\psi(G) \geq \frac{1}{q}\psi(C_n)$ or n is odd and $\psi(G) \geq \frac{1}{q+1}\psi(C_n)$, then G is solvable and either its Sylow p-subgroups or its Sylow q-subgroups are cyclic.

Proof We have $\frac{1}{q} \geq \frac{1}{2(q-1)}$ for $q \geq 2$, and even $\frac{1}{q+1} \geq \frac{1}{2(q-1)}$ if $q \geq 3$. Hence Theorem 8 applies in both cases.■

Finally we shall mention our other $\psi(G)$-based sufficient condition for solvability of finite groups.

Theorem 10

Let G be a finite group of order n satisfying

$$\psi(G) \geq \frac{3}{5} n\varphi(n).$$

Then G is solvable and $G'' \leq Z(G)$.

This condition is certainly not necessary for the solvability of G. For example, for $n = 8$ we have

$$\psi(C_2 \times C_2 \times C_2) = 15 < \frac{3}{5} n\varphi(n) = \frac{3}{5} \cdot 8 \cdot 4 = \frac{96}{5},$$

but the group $G = C_2 \times C_2 \times C_2$ is certainly solvable.

On the other hand, for $n = 60$, the simple group A_5 satisfies

$$\psi(A_5) = 211 > \frac{1}{5}n\varphi(n) = \frac{1}{5}60 \cdot 16 = 192.$$

Thus the statement: "If $\psi(G) \geq \frac{1}{5}n\varphi(n)$, then G is solvable" is incorrect. For the proof of Theorem 10 we used the following identity of Ramanujan (see [5]):

Theorem 11

If $q_1 = 2, q_2, \cdots, q_n, \cdots$ is the increasing sequence of **all** *primes, then*

$$\prod_{i=1,\cdots,\infty} \frac{q_i^2+1}{q_i^2-1} = \frac{5}{2},$$

What we needed was the following lemma:

Lemma 12

Let $p_1, p_2, \ldots, p_s$ be primes satisfying $p_1 < p_2 < \cdots < p_s$. If $p_1 > 3$ then

$$\prod_{i=1}^{s} \frac{p_i^2 - 1}{p_i^2 + 1} > \frac{5}{6}.$$

Proof Since $p_1 > 3$, Theorem 11 implies that

$$\frac{2^2+1}{2^2-1} \cdot \frac{3^2+1}{3^2-1} \prod_{i=1}^{s} \frac{p_i^2+1}{p_i^2-1} = \frac{5}{3} \cdot \frac{10}{8} \prod_{i=1}^{s} \frac{p_i^2+1}{p_i^2-1} < \frac{5}{2}.$$

Thus

$$\prod_{i=1}^{s} \frac{p_i^2+1}{p_i^2-1} < \frac{5}{2} \cdot \frac{3}{5} \cdot \frac{8}{10} = \frac{6}{5},$$

yielding

$$\prod_{i=1}^{s} \frac{p_i^2-1}{p_i^2+1} > \frac{5}{6},$$

as required. ■

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This is the END of my talk.

I would be happy to receive your comments.
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THANK YOU for your ATTENTION!