Calculations With Matrix Groups Over Rings And Applications To Arithmetic Groups

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Matrix Group Calculations

Matrix groups over commutative ring (here: $\mathbb{Z}$), given by (finite number) of generating matrices.

What can we say about such groups?

Over finite fields: ***matrix group recognition***

Uses: Divide-and-conquer approach. Data structure ***composition tree***. Reduction to simple groups.

Effective Homomorphisms, recursion to kernel, image.

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Finite Quotients key to computability

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Matrix Groups Over $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$

First consider $m=p^2$. ($m=p^a$ ditto iterated.)

Reduction mod p gives hom. φ: $\mathrm{SL}_n(\mathbb{Z}_m) \to \mathrm{SL}_n(\mathbb{Z}_p)$.

Kernel $\{I+pA \mid A \in \mathbb{Z}_p^{n\times n}\}$. Note: $\det(I+pA)=1+p\cdot\mathrm{Tr}(A)$.

Multiplication by addition of the A-parts modulo p.

$(I+pA)(I+pB)=I+p(A+B)+p^2\ldots\equiv I+p(A+B) \bmod m$

(Under map $A\mapsto I+pA$, $\ker\varphi$ is LIE-adjoint module)

Multiple primes: Subdirect product

```
gap> LoadPackage("matgrp"); # available for GAP 4.8.3
    […]
gap> g:=SL(3,Integers mod 1040);
SL(3,Z/1040Z)
gap> ff:=FittingFreeLiftSetup(g);;
gap> Size(g);
849852961151281790976000
gap> Collected(RelativeOrders(ff.pcgs));
[ [ 2, 24 ], [ 3, 1 ] ]
gap> m:=MaximalSubgroupClassReps(g);;time;
24631 #24 seconds
gap> List(m,x->Size(g)/Size(x));
[ 256, 7, 7, 8, 183, 183, 938119, 1476384, 3752476,
  123708, 123708, 123708, 31, 31, 3100, 3875, 4000 ]
```

Arithmetic Groups

Roughly: Discrete subgroup of Lie Group, defined by arithmetic properties on matrix entries(e.g. det=1, preserve form).

Definition: G linear algebraic group, over number field K. An ***arithmetic group*** is $\Gamma < G$, such that for integers $\mathcal{O}_K$ the intersection $\Gamma \cap G(\mathcal{O})$ has finite index in both intersectants.

Prototype: Subgroups of $SL_n(\mathbb{Z})$, $Sp_{2n}(\mathbb{Z})$ of finite index.

Applications: Number Theory (Automorphic Forms), Topology, Expander Graphs, String theory, ...

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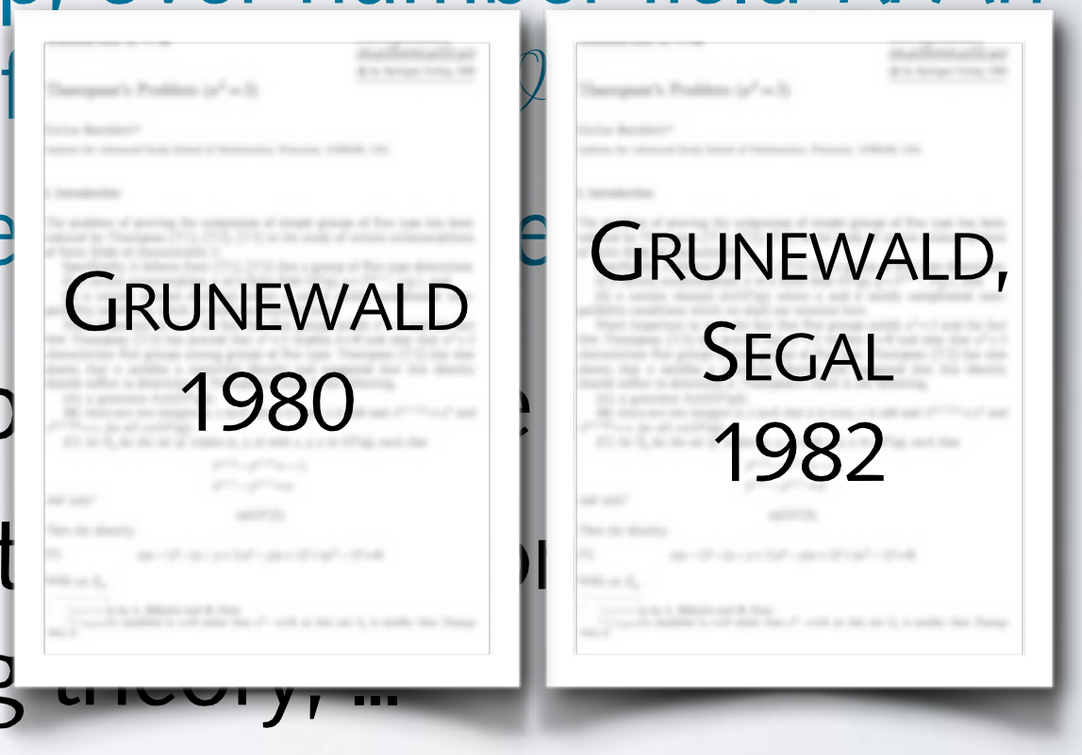

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Subgroups Of $SL_n(\mathbb{Z})$, $Sp_{2n}(\mathbb{Z})$

Take subgroup $G < SL_n(\mathbb{Z})$ (or Sp_{2n}) given by finite set of generators. G is arithmetic if it has finite index.

- Can we determine whether G has finite index?
- If G has finite index, can we determine it?

Here: Only SL case. SP similar. Others in work.

Joint work with ALLA DETINKO,
DANE FLANNERY
(St. Andrews / NUI Galway).

Subgroups Of $SL_n(\mathbb{Z})$, $Sp_{2n}(\mathbb{Z})$

Free subgroups, in general impossible, but Coset Enumeration may succeed in unbounded time.

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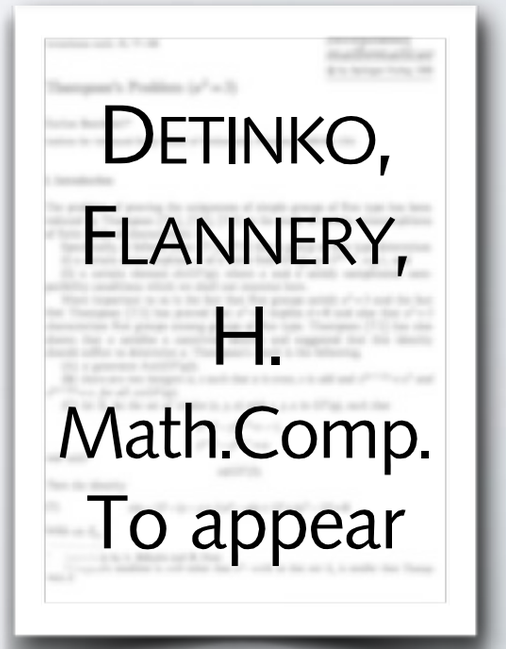
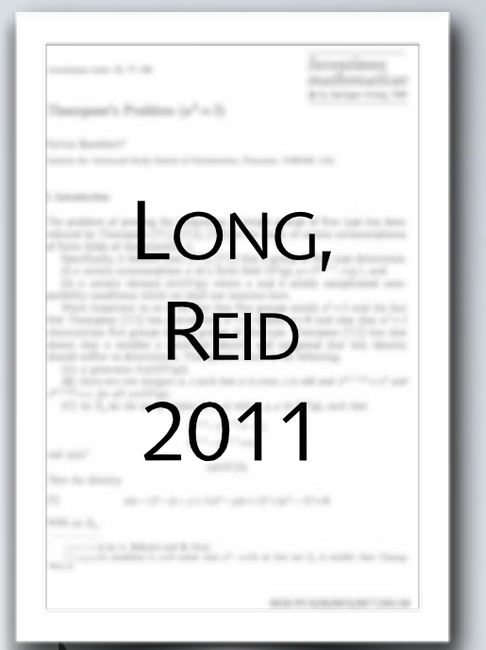


Easy Example



Let $SL_3(\mathbb{Z}) \geqq \beta_T =$

$$\left\langle \begin{pmatrix} -1+T^3 & -T & T^2 \\ 0 & -1 & 2T \\ -T & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ -T^2 & 1 & -T \\ T & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & T^2 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle,$$

then $[SL_3(\mathbb{Z}): \beta_{-2}]=3670016$.

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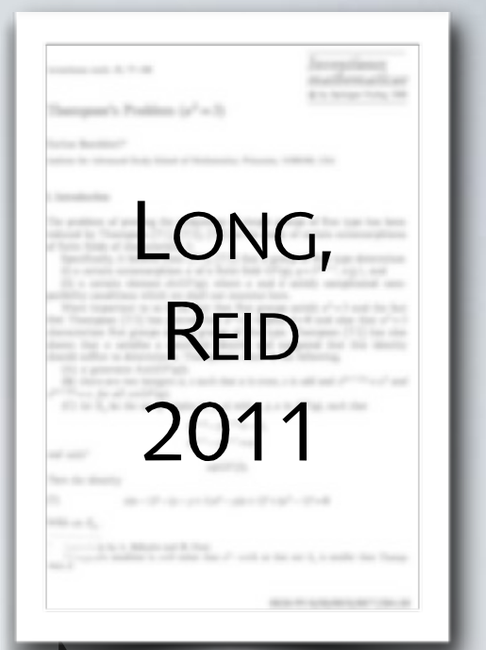

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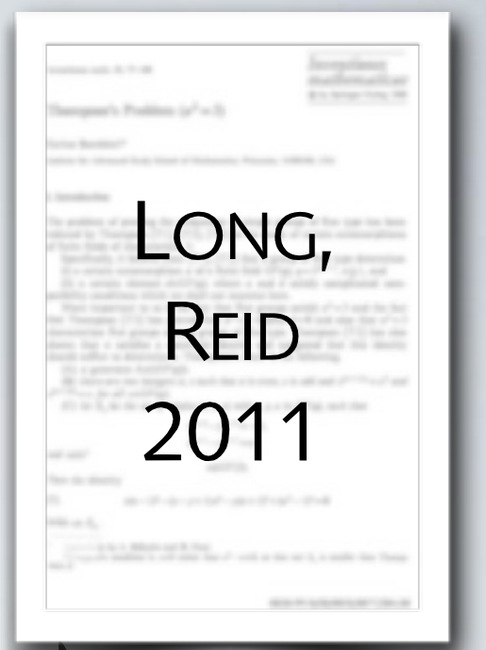

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then $[SL_3(\mathbb{Z}): \beta_{-2}] = 3670016$. (Barely) doable.

But $[SL_3(\mathbb{Z}): \beta_7] = 24193282798937316960$

$= 2^5 3^4 5 \cdot 7^{10} 19 \cdot 347821 \sim 2^{64}$. Hopeless.

New Approach

A subgroup of finite index defines a finite permutation quotient.

Use finite quotients, in particular congruence quotients, to determine the index?

Congruence Subgroups

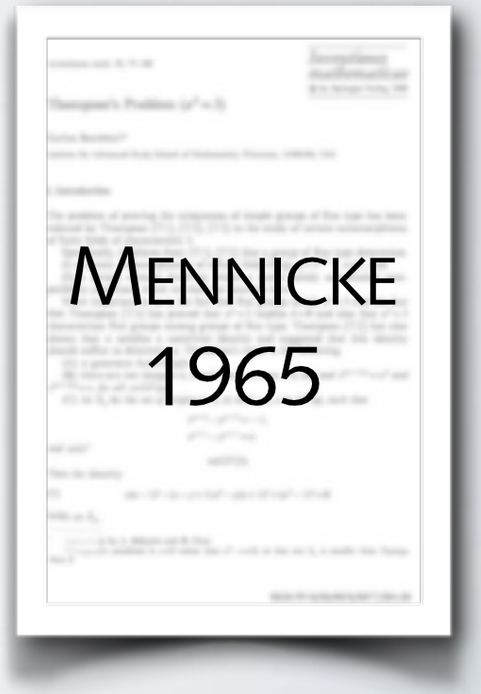
The m-th congruence subgroup $\Gamma_m \leq \mathrm{SL}_n(\mathbb{Z})$ is the kernel of the reduction φ_m modulo m. Image is $\mathrm{SL}_n(\mathbb{Z}_m)$.

If $G \leq \mathrm{SL}_n(\mathbb{Z})$ has finite index, there exists integer l such that $\Gamma_l \leq G$. The smallest such l is called the *level* of G.

Then $[\mathrm{SL}_n(\mathbb{Z}) : G] = [\mathrm{SL}_n(\mathbb{Z}_l) : \varphi_l(G)]$.

Calculate this second index from generators of G modulo l.

Thus sufficient to find level to get index.



Strategy

Consider congruence images $\varphi_m(G) < SL_n(\mathbb{Z}_m)$ for increasing values of m to find level l of G.

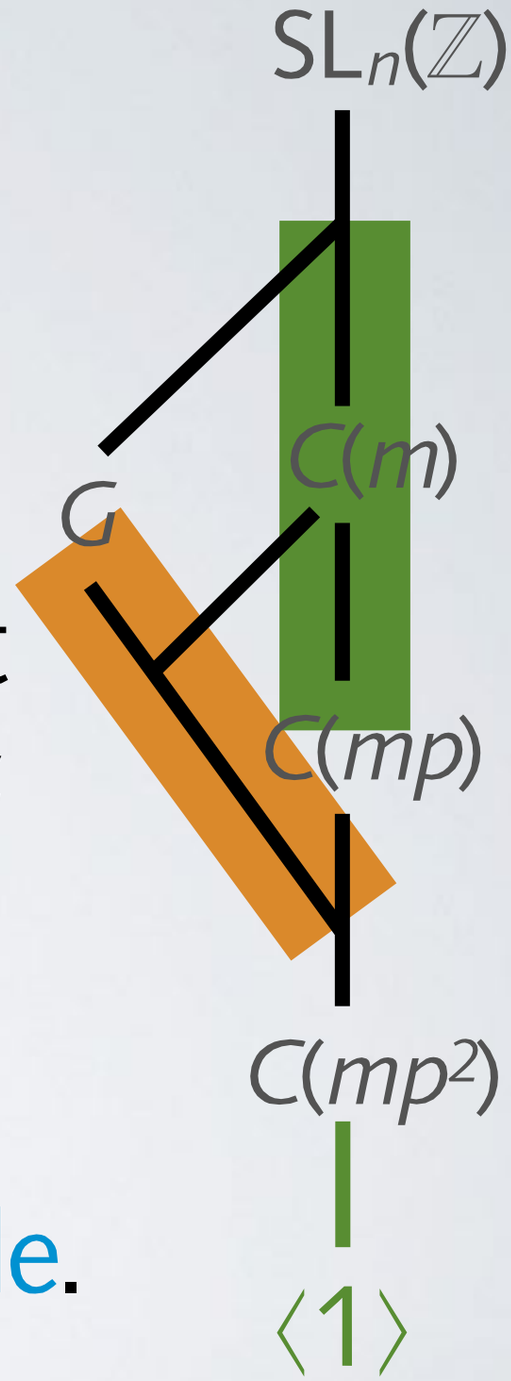
- Find the primes dividing l
- Find the prime powers dividing l
- Criterion on whether $I_m = [SL_n(\mathbb{Z}_m) : \varphi_m(G)]$ increases.

Same Index

Let $G \leq \mathrm{SL}_n(\mathbb{Z})$ and $C(m) = \ker \varphi_m$.

If for a given m and prime p we have that $I_m = I_{mp}$ but $I_{mp} \neq I_{mp^2}$, then (modulo mp^2) G contains a supplement to $C(mp)$.

We show such supplements typically do not exist, thus a stable index remains stable.



Kernel Supplements

Let p be prime, $a \geq 2$, $m=p^a$ and $H=\mathrm{SL}(n,\mathbb{Z}_m)$ for $n \geq 2$ (or $H=\mathrm{Sp}(2n, \mathbb{Z}_m)$ for $n \geq 1$). Let $C(k) \triangleleft H$ kernel mod k.

Theorem: (D-F-H.) $C(p^{a+1})$ has no proper supplement in $C(p^a)$.

Theorem: (Beisiegel 1977, Weigel 1995, …,D-F-H.)

Let $a=2$. $C(p)$ has a supplement in H if and only if

(a) $H=\mathrm{SL}(2,\mathbb{Z}_4)$, $\mathrm{SL}(2,\mathbb{Z}_9)$, $\mathrm{SL}(3,\mathbb{Z}_4)$, or $\mathrm{SL}(4,\mathbb{Z}_4)$.

(b) $H=\mathrm{Sp}(2,\mathbb{Z}_4)$, $\mathrm{Sp}(2,\mathbb{Z}_9)$.

Proof: Small cases/counterexample by explicit calculation. Use nice elements to show supplement contains kernel.

Index Algorithm

Assume that G has (unknown) finite index and level l. Assume we know the set $\mathscr{P}$ of primes dividing l.

1. Set $m=\mathrm{lcm}(4,\prod \mathscr{P})$.
2. While for any $p \in \mathscr{P}$ we have

   $[\mathrm{SL}_n(\mathbb{Z}_m):\varphi_m(G)]<[\mathrm{SL}_n(\mathbb{Z}_{pm}):\varphi_{pm}(G)]$, set $m:=pm$.
3. Repeat until index is stable, level divides m.

Show also that one can work prime-by-prime.

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2. While for any $p \in \mathscr{P}$ we have $[G:G^{(m)}] = [\mathrm{PSL}_n(\mathbb{Z}):\mathrm{PSL}_n(p)]$, set $m:=pm$.
3. Repeat until index is stable, level divides m.

A group projecting onto $\mathrm{PSL}_n(p)$ has only trivial subdirect products with subgroups of $\mathrm{PSL}_n(q)$

Show also that one can work prime-by-prime.

The Set Of Primes

Theorem: Let $n \geq 3$ and suppose G has finite index. The set $\mathscr{P}$ of primes dividing the level l of G consists of those primes p for which

1. $p>2$ and $G \bmod p \neq \mathrm{SL}_n(\mathrm{p})$, or

2. $p=2$ and $G \bmod 4 \neq \mathrm{SL}_n(\mathbb{Z}_4)$

Proof: If other primes divided the level, there would be a supplement modulo p^2 (or 8).

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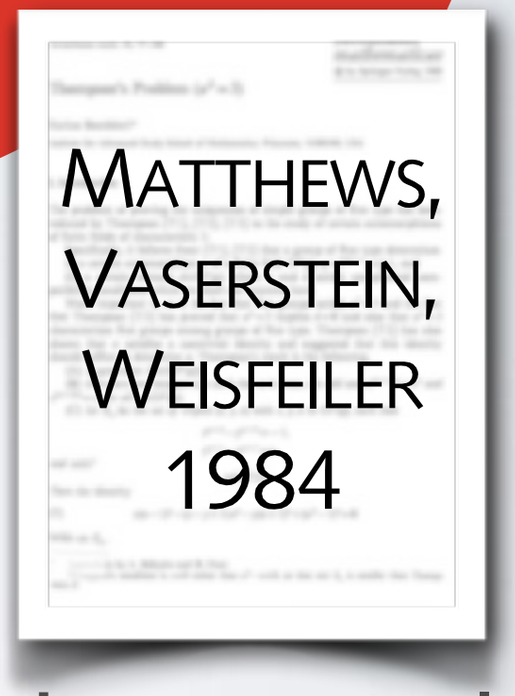
Zariski - density



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Finding Primes

We want primes for which $\varphi_p(G) \neq \mathrm{SL}_n(p)$. Methods:

a) Odd n, have ***transvection*** $t \in G$ (i.e. rk $(t-1)=1$).
   Let $N = \langle t \rangle^G$ normal closure. Primes for which $\varphi_p(t)$ not transvection or $\varphi_p(N)$ not abs. irr.

b) Test suitable set (possible b/c Steinberg rep.) of representations to remain abs. irr.

c) Eliminate possibilities of $\varphi_p(G)$ to lie in maximal subgroups of Aschbacher classes.
   So far done for small (prime) degrees.

Take $\mathbb{Z}$-lattice $L \leq \mathbb{Z}^{n\times n}$ spanned by G, rank n^2
Primes divide discriminant of L.

V

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E.g:. find suitable elements $a \in G$, whose image cannot lie in particular class:

- $|a| = \infty$, then $|\varphi_p(a)|$ divides m iff $a^m \equiv 1 \pmod{p}$
- $[a^m, b^m] \neq 1$ for $m = \exp(S_n)$, then $\varphi_p(G)$ monomial, iff $\varphi_p([a^m, b^m]) \equiv 1 \pmod{p}$

). Methods:

x (*t*-

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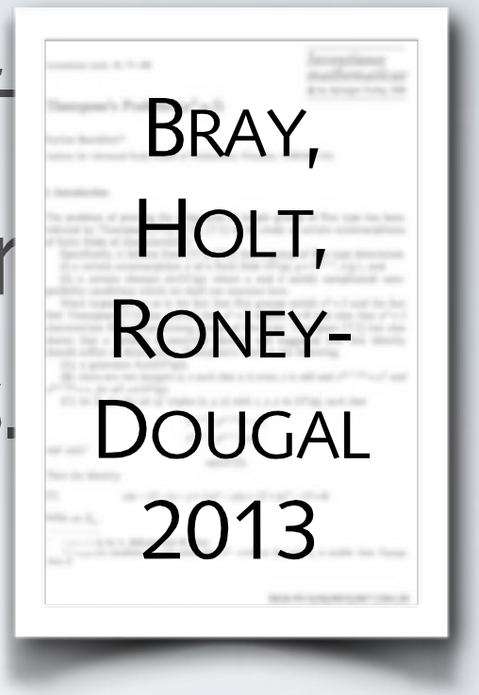
abs.

erg



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```
gap> g:=BetaT(7);
<matrix group with 3 generators>
gap> t:=b1beta(g); # transvection from Long/Reid paper
[ [-685,14,-98], [-16807,344,-2401], [2401,-49,344] ]
gap> IsTransvection(t);
1
gap> PrimesForDenseT(g,t,SL);time;
[ 7, 1021 ]
60
gap> MaxPCSPrimes(g,[7,1021],SL);time;
Try 7 7
Try 49 7
Try 343 7
Try 343 1021
Try 350203 1021
[350203, 2419328279893731696O ] #Proven Index in SL
291395 # about 5 minutes
```

```
gap> g:=Group([[778,2679,665],[323,797,665],
>       [6674504920,-1557328,340623049 49]],
>  [[-274290687,140904793,1960070592 ],[853,4560,294],
>       [151,930,209]]);;

gap> PrimesNonSurjective(g);  # about 2 sec.
#I  irrelevant prime 7
#I  Absolute irreducibility - found: [ 3, 5, 19 ] new:[ 19
#I  Monomial - found: [ 2, 3, 53 ] new:[ 53 ]
#I  Preserve a form - found: [ 3, 5 ] new:[  ]
#I  Element Order - found: [ 2, 3, 5, 19 ] new:[  ]
#I  Solvable - found: [ 2, 3, 5, 19, 53 ] new:[  ]
[ 2, 3, 5, 19, 53 ]

gap> MaxPCSPrimes(g,[2,3,5,19,53]); # about 25 sec.
Try 2,4,3,9,5,25,19,19^2,53, 53^2,30210 w. all primes
Index is 586082624189853029990 4=[ [ 2,13 ], [ 3,4 ],
 [ 13,3 ], [ 19,3 ], [ 31,1 ], [ 53,3 ], [ 127,1 ] ]
[ 30210, 586082624189853029990 4 ]
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Talk to me for more details!