The Strong Symmetric Genus of Almost All D-type Generalized Symmetric Groups

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Definition

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- The strong symmetric genus of the group G is denoted $\sigma^0(G)$.
- If $\sigma^0(G) > 1$ for a finite group G, then $\sigma^0(G) \geq 1 + \frac{|G|}{84}$.
- We have equality if G is a Hurwitz group.

Known results on the strong symmetric genus

- All groups G such that $\sigma^0(G) \leq 25$ are known.
  [Broughton, 1991; May and Zimmerman, 2000 and 2005; Fieldsteel, Lindberg, London, Tran and Xu, (Advised by Breuer) 2008]
- For each positive integer n, there is exists a finite group G with $\sigma^0(G) = n$.
  [May and Zimmerman, 2003]

Known results on the strong symmetric genus

The strong symmetric genus is known for the following groups:

- $PSL_2(q)$ [Glover and Sjerve, 1985 and 1987]
- $SL_2(q)$ [Voon, 1993]
- the sporadic finite simple groups [Conder, Wilson and Woldar, 1992; Wilson, 1993, 1997 and 2001]
- alternating and symmetric groups [Conder, 1980 and 1981]
- the hyperoctahedral groups [J, 2004]
- the remaining finite Coxeter groups [J, 2007]
- the generalized symmetric groups of type $G(n, 3)$ [J, 2010]

Generators and the Riemann-Hurwitz Equation

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  then we say that (x, y) is a (p, q, r) generating pair of G.

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- The existence of a (p, q, r) generating pair gives a faithful orientation preserving action of the group G on a surface S.

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  - The genus of the surface S is then found from the Riemann-Hurwitz formula:

$$\text{genus}(S) = 1 + \frac{|G|}{2}(1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}).$$

Minimal Generating Pairs

- A (p, q, r) generating pair of G is called a minimal generating pair if no generating pair for the group G gives an action on a surface of smaller genus.
- For the groups we will be working with $\sigma^0(G) \geq 2$ or equivalently any generating pair will be a (p, q, r) generating pair with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$.

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The Riemann-Hurwitz formula:

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A Lemma by Singerman

Lemma (Singerman)

Let G be a finite group such that $\sigma^0(G) > 1$. If $|G| > 12(\sigma^0(G) - 1)$, then G has a (p, q, r) generating pair with

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- Singerman's Lemma implies that if G has a minimal (p, q, r) generating pair such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \geq \frac{5}{6}$, then the strong symmetric genus is given by this generating pair.
- Since $\sigma^0(G) > 1$, we know that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$

More on Singerman's Lemma

- Recall: if G has a minimal (p, q, r) generating pair such that $\frac{5}{6} \leq \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, then the strong symmetric genus is given by this generating pair.
- The triples of numbers (p, q, r) that fit this requirement are:
  - $(2, 3, r)$ for any $r \geq 7$.
  - $(2, 4, r)$ for $5 \leq r \leq 11$.
  - $(3, 3, r)$ for $r = 4$ or $r = 5$.

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- The groups in this talk have S_n as a subgroup. So at least two numbers in the triple must be of even.
- The triples fitting both requirements are:
    - $(2, 3, r)$ for $r \geq 8$ even.
    - $(2, 4, r)$ for $5 \leq r \leq 11$.

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- $G(n, 1)$ is the symmetric group S_n.
- $G(n, 2)$ is the hyperoctahedral group B_n.
- The strong symmetric genus has been found for the groups:
  - $G(n, 1)$ [Conder, 1980]
  - $G(n, 2)$ and $G(n, 3)$ [J, 2004 and 2010]
  - $G(3, m)$, $G(4, m)$ and $G(5, m)$ [Ginter, Johnson, McNamara, 2008]

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- $D(n,m)$ is an index m subgroup of $G(n,m)$.
- $D(n,m)$ is the smallest group of $n \times n$ matrices containing
  - the permutation matrices and
  - the diagonal matrices with entries in a multiplicative cyclic group of size m each having determinant 1.
- The strong symmetric genus has been found for the groups $D(n,2)$ which are the finite Coxeter groups of type D [J, 2007]
- We will be looking at the groups $D(n,m)$ for $m > 2$.

Notation for elements of $D(n, m)$

- Recall that the group $D(n, m) = (\mathbb{Z}_m)^{n-1} \rtimes S_n$.
- An element of $D(n, m)$ will be denoted by $[\sigma, a]$ where
  - σ is an element of S_n, and
  - a is an element of $(\mathbb{Z}_m)^{n-1}$, which we will think of as a list of n integers modulo m such that the sum of the list is congruent to 0 modulo m.
- Notice that multiplication in the group is given by

$$[\sigma, a] \cdot [\tau, b] = [\sigma \cdot \tau, \tau^{-1}(a) + b]$$

where τ^{-1} is acting on the list a and the addition is term by term modulo m.

New generators from old

Suppose that S_n is generated by two elements σ and τ such that

- The number $m > 2$ divides the order of σ, and
- σ has two fixed points.
- If m and n are even then σ must have a third fixed point.

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Then $[\sigma, a]$ and $[\tau, b]$ generate $D(n, m)$ where

- b is a list of zeros,
- a is a list where one fixed point of σ has a 1 and the other fixed point has a -1,
- the rest of a is filled in so that the elements permuted by each cycle of σ add to zero modulo m and the elements permuted by each cycle of $\tau \cdot \sigma$ add to zero modulo m.

$3|m$, part I

Suppose that S_n is generated by two elements σ and τ such that

- $3|m$, $9 \not| m$, and the number $s = \frac{m}{3}$ divides the order of σ,
- τ has order 3, and
- both σ and τ have two fixed points.
- If m and n are even then σ must have a third fixed point.

$3|m$, part II

Then $[\sigma, a]$ and $[\tau, b]$ generate $D(n, m)$ where

- a is a list where one fixed point of σ has a 3 and the other fixed point has a -3,
- b is a list where one fixed point of τ has an s and the other has a number $-s$, and
- the rest of a and b are filled in so that each of the following add to 0 modulo m:
    - the elements of a permuted by each cycle of σ
    - the elements of b permuted by each cycle of τ, and
    - the elements of $\sigma^{-1}(b) + a$ permuted by each cycle of $\tau \cdot \sigma$ add to zero modulo m.

Orders

- Given the σ and τ that generate S_n and satisfy the conditions from either of the past two slides
- the new elements that we created $[\sigma, a]$ and $[\tau, b]$ generate $D(n, m)$.
- In addition the orders of $[\sigma, a]$,$[\tau, b]$ and

$$[\tau, b] \cdot [\tau, b] = [\tau \cdot \sigma, \sigma^{-1}(b) + a]$$

are the same as σ, τ and $\tau \cdot \sigma$, respectively.

Function

Given an integer $m > 2$ define $r(m)$ using the following criteria:

- If $m = 3, 4$, or 6, then $r(m) = 8$
- If $m = 12$, then $r(m) = 12$.
- If $3|m$ but $9 \not| m$ then
  let $r(m) = \frac{m}{3}$ for m even and $r(m) = \frac{2m}{3}$ for m odd.
- Otherwise let $r(m) = m$ for m even and $r(m) = 2m$ for m odd.

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Notice that

- for all m, $m|3r(m)$,
- if $3 \not| m$ or $9|m$, then $m|r(m)$, and
- $r(m)$ is always even.

Conder's Generators

We use Conder's Papers "More on generators for alternating and symmetric groups" Quart. J. Math. Oxford (2), 32 (1981) 137-163.

Using the coset diagrams from the paper, we see that given $m > 2$ there are generators σ and τ for all but finitely many symmetric groups S_n such that

- σ has order $r(m)$,
- τ has order 3,
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- σ has order $r(m)$,
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- σ has three fixed points, and τ has two fixed points.

For a fixed m, this allows for the creation of a $(2, 3, r(m))$ generating pair for all but finitely many $D(n, m)$.

We are left to show that these generators are a minimal generating pair.

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Other Generators

- To claim that our generators are a minimal generating pair, we need to show that there cannot be a generating pair with a better (p, q, r) triple.
- If any prime power p^i which divides m does not divide q or r, then $D(n, m)$ cannot have a $(2, q, r)$ generating pair.
- The best (hyperbolic) triple not of the form $(2, q, r)$ where two of the three numbers are even is $(3, 4, 4)$.
- Notice that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{r(m)} > \frac{5}{6} = \frac{1}{3} + \frac{1}{4} + \frac{1}{4}.$$

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- Checking sums of reciprocals leaves two cases,
  - $m = 20$ and the triple $(2, 4, 5)$, and
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- Checking sums of reciprocals leaves two cases,
  - $m = 20$ and the triple $(2, 4, 5)$, and
  - $m = 28$ and the triple $(2, 4, 7)$.
- It turns out that in these two cases the $(2, 4, r)$ triple has a generating pair for all but finitely many cases.

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- This paper does not consider the case $(2, 4, 5)$ since that work had been done earlier by Graham Higman.
- With a slight modification to the coset diagrams in this paper and a similar process to what we did in the $(2, 3, r(m))$ case, we create a $(2, 4, 7)$-generating pair for all but finitely many of the $D(n, 28)$ groups.

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- This leaves just the case where $m = 20$.
- The coset diagrams for the $(2, 4, 5)$-generating pairs for all but finitely many of the groups S_n was unpublished work.
- Therefore we created our own collection of coset diagrams which give appropriate generators for all but finitely many S_n.
- As in earlier cases this $(2, 4, 5)$-generating pair of S_n can be modified to be a $(2, 4, 5)$-generating pair of $D(n, 20)$

Theorem

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Given a fixed $m > 2$, where m is neither 20 or 28, for all but finitely many positive integers n, the D-type generalized symmetric group $D(n, m)$ has a $(2, 3, r(m))$-minimal generating pair.
In addition all but finitely many of the groups $D(n, 20)$ have a $(2, 4, 5)$-minimal generating pair and all but finitely many of the groups $D(n, 28)$ have a $(2, 4, 7)$-minimal generating pair.

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Given a fixed $m > 2$, where m is neither 20 or 28, for all but finitely many positive integers n

$$\sigma^0(D(n,m)) = \frac{n!m^{n-1}(r(m)-6)}{12r(m)} + 1.$$

In addition for all but finitely many positive integers n

$$\sigma^0(D(n,20)) = \frac{n!m^{n-1}}{40} + 1 \ and \ \sigma^0(D(n,28)) = \frac{3n!m^{n-1}}{56} + 1.$$