Non-cancellation group of a direct product

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Outline of Talk

- Introduction
- Non cancellation set of χ_o-groups
- Non cancellation set of $\mathcal{K}$-groups
- The category $\mathrm{Grp_F}$
- Non cancelation group of a direct product

Abstract

Consider the semidirect product $G_i = \mathbb{Z}_{n_i} \rtimes_{\omega_i} \mathbb{Z}$. Methods for computation of the non-cancellation groups $\chi(G_1 \times G_2)$ and $\chi(G_i^k)$, $k \in \mathbb{N}$ were developed in literature. We develop a general method of computing $\chi(G_1 \times G_2, h)$, where $h : F \hookrightarrow G_1 \subseteq G_1 \times G_2$ and F a finite group.

Introduction

- N f.g. nilpotent group
- $\mathcal{G}(N)$: isomorphism classes of f.g nilpotent groups M such that for every prime p $M_p \simeq N_p$
- $\mathcal{G}(N)$ has a group structure
- χ_o : class of all G f.g. groups with $[G, G]$ finite
- Non cancellation set $\chi(G)$:
  set of all isomorphism classes of groups H such that $G \times \mathbb{Z} \cong H \times \mathbb{Z}$
- N is a f.g nilpotent group $[N, N]$ finite then $\mathcal{G}(N) = \chi(N)$
- $\chi(G)$ has a group structure similar to the group structure on the Mislin genus of a nilpotent χ_o- group
- $H \times \mathbb{Z} \cong G \times \mathbb{Z}$ iff. $H_\pi \cong G_\pi$ (π for every finite set of primes π
- χ_o- group H, $\chi(H)$ coincides with the restricted genus $\Gamma_f(H)$ of H

Introduction 2 continued....

- Fix F finite grp. $h : F \longrightarrow G$ be a monomorphism
- $\mathrm{Grp_F}$: category of groups under F
- objects: (G, h), (G_1, k) group homomorphisms $h : F \longrightarrow G$ and $k : F \longrightarrow G_1$
- morphisms: group homomorphism $\beta : G \longrightarrow G_1$ such that $\beta \circ h = k$
- π-localization of an object: $h : F \longrightarrow G$ is the object $h_\pi : F \longrightarrow G_\pi$ where π is set of primes
- $\mathcal{X}_F$: full subcategory of χ_o- groups
- $\Gamma_f(h)$: restricted genus: isomorphism classes k such that k_π is isomorphic to h_π for $k \in \mathcal{X}_F$
- F is a trivial group then $\mathcal{X}_F$ is identified with χ_o-groups
- F trivial : $\chi(G, h)$, non cancellation group of G under F coincides with $\chi(G)$

Introduction 3 continued....

- $\mathcal{K}$ be the class of groups: $T \rtimes_{\omega} F$, F finite rank free abelian group T finite abelian group
- $H = G(m, u) = \mathbb{Z}_m \rtimes_{\nu} \mathbb{Z}$, $(m, u) = 1$.
- H is a $\mathcal{K}$- group
- $\mathcal{K}_{\mathrm{F}}$ full subcategory of $\mathrm{Grp}_{\mathrm{F}}$

Problem Statement

- Let $G_i = \mathbb{Z}_{m_i} \rtimes_{\omega_i} \mathbb{Z}$ and let $h : F \hookrightarrow G_1 \times G_2$ where F is a finite group and h is a monomorphism. We develop a general method for computing $\chi(G_1 \times G_2, h)$.

Non cancellation set of χ_o-groups

- χ_o- group G: assign a natural number $n(G) = n_1 n_2 n_3$
  - n_1 : exponent of T_G
  - n_2 : exponent of $\mathrm{Aut}(T_G)$
  - n_3 : exponent of the torsion of the center T_{Z_G}
- G χ_o-group with $[G : H] < \infty$, ; $T_G = T_H$
- $W = \{H < G, ([G : H], n) = 1\}$ then $H \times \mathbb{Z} \cong G \times \mathbb{Z}$.
- K group, $K \times \mathbb{Z} \cong G \times \mathbb{Z}$,then exists $J \in W$ with $J \cong K$.
- Given $J_1, J_2 \in W$, if $[G : J_1] \equiv \pm 1[G : J_2] \mod n$, then $J_1 \cong J_2$
- non-cancellation set $\chi(G)$ coincides with the (finite) set $\{[J] : J \in W\}$.
- $\mathbb{Z}_n^*/\{1, -1\} \longrightarrow \chi(G)$ induces a group structure on $\chi(G)$
- G nilpotent, $\chi(G)$ coincides with Hilton-Mislin Genus.

Non cancellation set of $\mathcal{K}$-groups

- $G = \mathbb{Z}_m \rtimes_\omega \mathbb{Z}$
- $\omega : \mathbb{Z} \longrightarrow \mathrm{Aut}(\mathbb{Z}_m)$ automorphism of $\mathbb{Z}_m$ defined $\omega(1)(t) = ut$
- d : multiplicative order of u modulo m
- $\chi(G) = \mathbb{Z}_d^*/\{1, -1\}$

Non cancellation group of a morphism $\chi(G,h)$

- $n = n(G)$, $X(n) = \{u \in \mathbb{N} : (u,n) = 1\}$
- $Y(G,h) =$
  $\{u \in X(n), ; \text{exist } K < G, ; [G:K] = u, \ (K, h_K) \text{ member } \chi(G,h)\}$
- $G_u < G, \ ; T_G \subseteq G_u$ and $[G : G_u] = u$ for each $u \in Y(G,h)$
- $h_u : x \mapsto h(x)$ induced homomorphism
- $\varsigma : Y(G,h) \longrightarrow \chi(G,h)$
- $Y^*(G,h)$:image of $Y(G,h)$ in $\mathbb{Z}_n^*$, $Y^*(G,h) < \mathbb{Z}_n^*$
- G χ_o-group, $h : F \to G$ monomorphism, F finite group
- $\varsigma : Y(G,h)/\pm 1 \longrightarrow \chi(G,h)$ induces a group structure on $\chi(G,h)$
- F trivial, $\chi(G,h) = \chi(G)$

Non cancellation group of a morphism of an object in $\mathcal{K}_F$

- $G = \mathbb{Z}_m \rtimes_\omega \mathbb{Z}$
- $t = d(G)$: smallest invariant factors of $\mathrm{Im}\,\omega$
- $V(G,h) = \{u \in X(t)\}$ s.t $K < G$, $[G:K] = u$ and (K, h_K) is a member of $\chi(G,h)$
- $V^*(G,h)$ be the image of $V(G,h)$ in $\mathbb{Z}_t^*$
- Choose $K < G$ of G s.t. (K, h_K) member in $\chi(G,h)$ and $[G:K] = u$
- $t \mid n(G)$

Proposition

Let $n = n(G)$ *and* $\rho : Y(G,h) \longrightarrow \chi(G,h)$ *be the epimorphism that takes a residue* $\mathrm{mod}\, \mathrm{n}$ *and reduces it* $\mathrm{mod}\ \mathrm{t}$*. The epimorphism* $\zeta : Y^*(G,h) \longrightarrow \chi(G,h)$ *factorises through the epimorphism* ξ'*:*

$$\begin{tikzcd} Y^*(G,h) \arrow[r, "\zeta"] \arrow[d, "\rho"'] & \chi(G,h) \\ V^*(G,h) \arrow[ur, "\xi'"'] & \end{tikzcd}$$

Theorem

Theorem

For $m \in V(G, h)$, the following conditions are equivalent:

(a) $\overline{m} \in \ker[V^*(G, h) \longrightarrow \chi(G, h)]$.

(b) *There exists* $\alpha \in \mathrm{Aut}\,(T)$ *with* $v \circ h = h$ *such that* $\alpha \in N_{\mathrm{Aut}\,T} \,\mathrm{Im}\,\omega$ *and for an automorphism* $\bigwedge : \mathrm{Im}\,\omega \longrightarrow \mathrm{Im}\,\omega$ *defined by* $v \mapsto \alpha\, v \alpha^{-1}$, *we have* $\det(\bigwedge) = \pm\overline{m}^{-1} \in V^*(G, h)$.

Non cancelation group of $(G_1 \times G_2, h)$

- $G_i = G(m_i, u_i)$ with $\gcd(m_1, m_2) = 1,\ \gcd(m_i, u_j) = 1,\ i, j = 1, 2$
- d_i: multiplicative order of u_i modulo m_i
- $d = \operatorname{lcm}(d_1, d_2)$ and $m = \operatorname{lcm}(m_1, m_2)$.
- If $\gcd(d_1, d_2) = 1$ let $t = d_1 d_2$ and if $\gcd(d_1, d_2) \neq 1$ let $t = \gcd(d_1, d_2)$
- $G_1 \times G_2 = (\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}) \rtimes_\omega \mathbb{Z}^2$
- $\omega : \mathbb{Z}^2 \longrightarrow \operatorname{Aut}(\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2})$
- $\omega(\epsilon_i) = \omega_i : (t_1, t_2) \mapsto (u_1^{\delta(i,1)} t_1, u_2^{\delta(i,2)} t_2)$
- $\{\epsilon_1, \epsilon_2\}$ standard basis of $\mathbb{Z}^2$
- $J = \operatorname{Im} \omega = < \omega_1, \omega_2 >$ is a free $\mathbb{Z}_d$-module

Non cancelation group of $(G_1 \times G_2, h)$ continued...

- $\sigma \in \mathrm{End}(J)$, $\det(\sigma) \in \mathbb{Z}_d$
- $J_* = N_{\mathrm{Aut}(T_{G_1 \times G_2})} J$
- $\alpha \in \mathrm{Aut}(T_{G_1 \times G_2})$, $\bigwedge_\alpha$ inner automorphism of J: $\bigwedge_\alpha : v \mapsto \alpha\, v\, \alpha^{-1}$
- q_2 be a multiple of q such that $q_2 q$ divides m.
- $e_1 = (1, 0)$, $e_2 = (0, 1)$:elements of $T_{G_1 \times G_2}$
- $F = \{a q_2 e_2 \; : \; a \in \mathbb{Z}\}$ $F < T_{G_1 \times G_2}$
- $h : F \hookrightarrow G_1 \times G_2$ inclusion map

An inner automorphism of $\mathrm{Aut}(\mathrm{T}_{\mathrm{G}_1 \times \mathrm{G}_2})$

- Fix $\alpha \in J^*$ such that $\alpha(x) = x \ \forall x \in F$
- exists a 2×2 matrix (α_{ij}) of integers such that $\alpha(e_i) = \sum_{j=1}^{2} \alpha_{ji} e_j$
- Suppose that $\bigwedge_\alpha$ is the inner automorphism of J determined by α.

Proposition

For the matrix (α_{ij}), α_{ii} is a unit modulo m for $i = 1, 2$.

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Proof

- $\alpha(0, q_2) = (0, q_2)$ since $(0, q_2) \in F$
- $\alpha(0, q_2) = (q_2\alpha_{12}, q_2\alpha_{22}) = (0, q_2)$
- m divides α_{12}
- q divides α_{12}, while α_{22} is a unit modulo m
- $[\alpha] = \begin{pmatrix} \alpha_{11} & tq \\ \alpha_{21} & u \end{pmatrix}$
- $\det(\alpha) = \alpha_{11}u - \alpha_{21}tq$
- Claim: α_{11} is a unit modulo m
- α_{11} is not a unit modulo m
- let p common prime divisor of α_{11} and m
- p divides q and p divides $\det(\alpha)$ contradiction!!! ($\det(\alpha)$ is a unit modulo m) (α is an automorphism).
- Thus α_{11} is a unit modulo m.

Proposition B

Proposition

The inner automorphism $\bigwedge_\alpha$ of J coincides with the identity automorphism of J.

Proof

- for $\bigwedge_\alpha$ exists a matrix $(\bigwedge_{ij})$ of integers such that $\bigwedge \omega_i = \omega_1^{\bigwedge_{1i}} \omega_2^{\bigwedge_{2i}}$ for each i
- let $\bigwedge \omega_i = v_i$. Also $\bigwedge \omega_i = \alpha \omega_i \alpha^{-1} = v_i \iff \alpha \omega_i = v_i \alpha$
- $\alpha \omega_i (e_i) = \alpha(u_i e_i) = \sum_{j=1}^{2} u_i^{\delta(i,j)} \alpha_{ji} e_j$
- $v_i \alpha(e_i) = v_i \left(\sum_{j=1}^{2} \alpha_{ji} e_j \right) = \sum_{j=1}^{2} \alpha_{ji} u_i^{\delta(i,j) \bigwedge_{ji}} e_j$
- $\sum_{j=1}^{2} u_i^{\delta(i,j)} \alpha_{ji} e_j = \sum_{j=1}^{2} \alpha_{ji} u_i^{\delta(i,j) \bigwedge_{ji}} e_j$
- $j = i$
    - α_{ii} is a unit modulo m
    - $u_i^{\bigwedge_{ii}} \equiv u_i \bmod m$
    - $\bigwedge_{ii} \equiv 1 \bmod d$
    - $\bigwedge_{ii} \equiv 1 \bmod t, \ (t \mid d)$

Proof continued....

- $j \neq i$
  - $\alpha\omega_i(e_j) = \alpha(u^{\delta(i,j)}(e_j)) = \alpha(e_j) = \sum_{k=1}^{2} \alpha_{kj} e_k$
  - $v_i\alpha(e_j) = \sum_{k=1}^{2} \alpha_{kj} u_k^{\bigwedge_{ki}} e_k$
  - Therefore $\sum_{k=1}^{2} \alpha_{kj} e_k = \sum_{k=1}^{2} \alpha_{kj} u_k^{\bigwedge_{ki}} e_k$
  - $k = j$. Since α_{jj} is a unit modulo m
  - then $u_j^{\bigwedge_{ji}} \equiv 1 \mod m$, that is, $\bigwedge_{ji} \cong 0 \mod d$
  - Consequently $\bigwedge_{ji} \equiv 0 \mod t$. Thus $\det(\bigwedge_{\alpha}) = 1$
  - $\bigwedge_{\alpha}$ is identity on J

Proposition C

Proposition

The pair $(G_1 \times G_2, h)$ *is an object of* $\mathcal{K}_F$ *for which* $(G_1 \times G_2, h) = \mathbb{Z}_t^*/\pm 1$.

- Proof
  - Proposition follows from Theorem 1 and Proposition 3

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