Small doubling properties in orderable groups

Patrizia LONGOBARDI

UNIVERSITÀ DEGLI STUDI DI SALERNO

Groups St Andrews 2017

in Birmingham

5th-13th August 2017

Basic definition

Definition

If S, T are finite sets of integers, then we put

$$S + T := \{x + y \mid x \in S, y \in T\}, 2S := \{x_1 + x_2 \mid x_1, x_2 \in S\} .$$

$S + T$ is also called the (Minkowski) sumset of S and T.

If $S = \{x\}$, then we denote $S + T$ by $x + T$ and if $T = \{y\}$, then we write $S + y$ instead of $S + \{y\}$.

Questions

What can be said about $2S$ if we know some property of S ?

What can be said about S if we have some bound for $|2S|$?

Basic definition

Definition

If S, T are finite sets of integers, then we put

$$S + T := \{x + y \mid x \in S, y \in T\}, 2S := \{x_1 + x_2 \mid x_1, x_2 \in S\} .$$

$S + T$ is also called the (Minkowski) sumset of S and T.

If $S = \{x\}$, then we denote $S + T$ by $x + T$ and if $T = \{y\}$, then we write $S + y$ instead of $S + \{y\}$.

Questions

What can be said about $2S$ if we know some property of S ?

What can be said about S if we have some bound for $|2S|$?

Background

Remark 1

Let S be a finite set of integers with k elements. Then

$$|2S| \geq 2k - 1.$$

Proof. Let $S = \{x_1, x_2, \cdots, x_k\}$, and assume $x_1 < x_2 < \cdots < x_k$.

Clearly

$$2x_1 < x_1 + x_2 < 2x_2 < x_2 + x_3 < 2x_3 < \cdots < 2x_{k-1} < x_{k-1} + x_k < 2x_k$$

and each of these elements belongs to $2S$. Hence $|2S| \geq 2k - 1$, as required. //

Background

Remark 1

Let S be a finite set of integers with k elements. Then

$$|2S| \geq 2k - 1.$$

Proof. Let $S = \{x_1, x_2, \cdots, x_k\}$, and assume $x_1 < x_2 < \cdots < x_k$.

Clearly

$$2x_1 < x_1 + x_2 < 2x_2 < x_2 + x_3 < 2x_3 < \cdots < 2x_{k-1} < x_{k-1} + x_k < 2x_k$$

and each of these elements belongs to $2S$. Hence $|2S| \geq 2k - 1$, as required. //

Background

Remark 1

Let S be a finite set of integers with k elements. Then

$$|2S| \geq 2k - 1.$$

Proof. Let $S = \{x_1, x_2, \cdots, x_k\}$, and assume $x_1 < x_2 < \cdots < x_k$.

Clearly

$$2x_1 < x_1 + x_2 < 2x_2 < x_2 + x_3 < 2x_3 < \cdots < 2x_{k-1} < x_{k-1} + x_k < 2x_k$$

and each of these elements belongs to $2S$. Hence $|2S| \geq 2k - 1$, as required. //

Background

Remark 2

Let S be a finite set of integers with k elements.

If S is an **arithmetic progression**:

$$S = \{a, a+r, a+2r, \cdots, a+(k-1)r\},$$

then

$$|2S| = 2k-1.$$

Proof. We have

$$2S = \{2a, 2a+r, 2a+2r, ..., 2a+(2k-2)r\}.$$

//

Background

Remark 2

Let S be a finite set of integers with k elements.

If S is an **arithmetic progression**:

$$S = \{a, a+r, a+2r, \cdots, a+(k-1)r\},$$

then

$$|2S| = 2k-1.$$

Proof. We have

$$2S = \{2a, 2a+r, 2a+2r, ..., 2a+(2k-2)r\}.$$

//

Background

Remark 2

Let S be a finite set of integers with k elements.

If S is an **arithmetic progression**:

$$S = \{a, a + r, a + 2r, \cdots, a + (k-1)r\},$$

then

$$|2S| = 2k - 1.$$

Proof. We have

$$2S = \{2a, 2a + r, 2a + 2r, ..., 2a + (2k-2)r\}.$$

//

Background

Remark 3

Let S be a finite set of integers with k elements.

If $|2S| = 2k - 1$, then S is an arithmetic progression.

Proof. Let $S = \{x_1, x_2, \cdots, x_k\}$, and assume $x_1 < x_2 < \cdots < x_k$. Then $2S = \{2x_1, x_1 + x_2, 2x_2, x_2 + x_3, 2x_3, \cdots, 2x_{k-1}, x_{k-1} + x_k, 2x_k\}$ with $2x_1 < x_1 + x_2 < 2x_2 < x_2 + x_3 < 2x_3 < \cdots < 2x_{k-1} < x_{k-1} + x_k < 2x_k$.

Clearly $x_2 = x_1 + (x_2 - x_1)$.

It holds $2x_1 < x_1 + x_3 < 2x_3$ with $x_1 + x_3 \neq x_1 + x_2, x_2 + x_3$. Therefore $x_1 + x_3 = 2x_2$ and $x_3 = 2x_2 - x_1 = x_2 + (x_2 - x_1)$. Analogously $x_2 + x_4 = 2x_3$ and $x_4 = 2x_3 - x_2 = x_3 + (x_3 - x_2) = x_3 + (x_2 - x_1)$, and so on. //

Background

Remark 3

Let S be a finite set of integers with k elements.

If $|2S| = 2k - 1$, then S is an arithmetic progression.

Proof. Let $S = \{x_1, x_2, \cdots, x_k\}$, and assume $x_1 < x_2 < \cdots < x_k$. Then $2S = \{2x_1, x_1 + x_2, 2x_2, x_2 + x_3, 2x_3, \cdots, 2x_{k-1}, x_{k-1} + x_k, 2x_k\}$ with $2x_1 < x_1 + x_2 < 2x_2 < x_2 + x_3 < 2x_3 < \cdots < 2x_{k-1} < x_{k-1} + x_k < 2x_k$.
Clearly $x_2 = x_1 + (x_2 - x_1)$.
It holds $2x_1 < x_1 + x_3 < 2x_3$ with $x_1 + x_3 \neq x_1 + x_2, x_2 + x_3$. Therefore $x_1 + x_3 = 2x_2$ and $x_3 = 2x_2 - x_1 = x_2 + (x_2 - x_1)$. Analogously $x_2 + x_4 = 2x_3$ and $x_4 = 2x_3 - x_2 = x_3 + (x_3 - x_2) = x_3 + (x_2 - x_1)$, and so on. //

Remark 3

Let S be a finite set of integers with k elements.

If $|2S| = 2k - 1$, then S is an arithmetic progression.

Proof. Let $S = \{x_1, x_2, \cdots, x_k\}$, and assume $x_1 < x_2 < \cdots < x_k$. Then $2S = \{2x_1, x_1 + x_2, 2x_2, x_2 + x_3, 2x_3, \cdots, 2x_{k-1}, x_{k-1} + x_k, 2x_k\}$ with $2x_1 < x_1 + x_2 < 2x_2 < x_2 + x_3 < 2x_3 < \cdots < 2x_{k-1} < x_{k-1} + x_k < 2x_k$.
Clearly $x_2 = x_1 + (x_2 - x_1)$.
It holds $2x_1 < x_1 + x_3 < 2x_3$ with $x_1 + x_3 \neq x_1 + x_2, x_2 + x_3$. Therefore $x_1 + x_3 = 2x_2$ and $x_3 = 2x_2 - x_1 = x_2 + (x_2 - x_1)$. Analogously $x_2 + x_4 = 2x_3$ and $x_4 = 2x_3 - x_2 = x_3 + (x_3 - x_2) = x_3 + (x_2 - x_1)$, and so on. //

Additive Number Theory

Direct and Inverse theorems

Gregory A. Freiman,
Foundations of a structural theory of set addition
Translations of mathematical monographs, **37**, American Mathematical Society, Providence, Rhode Island, 1973.

Starting point

Additive Number Theory

Direct and Inverse theorems

Gregory A. Freiman,
Foundations of a structural theory of set addition
Translations of mathematical monographs, **37**, American Mathematical Society, Providence, Rhode Island, 1973.

Additive Number Theory

Direct and Inverse theorems

M.B. Nathanson
Additive number theory - Inverse problems and geometry of sumsets
Springer, New York, 1996.

A. Geroldinger, I.Z. Ruzsa,
Combinatorial Number Theory and Additive Group Theory
Birkäuser, Basel - Boston - Berlin, 2009.

Gregory A. Freiman, *Structure theory of set addition*, Astérisque, **258** (1999), 1-33

"Thus a **direct problem** *in additive number theory is a problem which, given summands and some conditions, we discover something about the set of sums. An* **inverse problem** *in additive number theory is a problem in which, using some knowledge of the set of sums, we learn something about the set of summands."*

Background - Direct and Inverse problems

Gregory A. Freiman, *Structure theory of set addition*, Astérisque, **258** (1999), 1-33

"Thus a **direct problem** *in additive number theory is a problem which, given summands and some conditions, we discover something about the set of sums. An* **inverse problem** *in additive number theory is a problem in which, using some knowledge of the set of sums, we learn something about the set of summands."*

Background

Let S be a finite set of integers with k elements, and

$$2S = \{x + y \mid x, y \in S\}.$$

Then $$|2S| \geq 2k - 1$$

and $$|2S| = 2k - 1 \text{ if and only if } S \text{ is an arithmetic progression.}$$

Questions

What can be said about S if $|2S|$ is not much greater than this minimal value?

What is the structure of S if $|2S| \leq \alpha k$, where α is any given positive number?

Background

Let S be a finite set of integers with k elements, and

$$2S = \{x + y \,|\, x, y \in S\}.$$

Then
$$|2S| \geq 2k - 1$$

and
$$|2S| = 2k - 1 \text{ if and only if } S \text{ is an arithmetic progression.}$$

Questions

What can be said about S if $|2S|$ is not much greater than this minimal value?

What is the structure of S if $|2S| \leq \alpha k$, where α is any given positive number?

Let S be a finite set of integers.

Question

Determine the **structure** of S if $|2S|$ satisfies

$$|2S| \leq \alpha|S| + \beta$$

for some small $\alpha \geq 1$ and small $|\beta|$.

Problems of this kind are called

inverse problems of small doubling type.

Background

G.A. Freiman,
On the addition of finite sets I,
Izv. Vyss. Ucebn. Zaved. Matematika **6** (13) (1959), 202-213.

G.A. Freiman,
Inverse problems of additive number theory IV. On the addition of finite sets II,
(Russian) Elabuž. Gos. Ped. Inst. Učen. Zap., **8** (1960), 72-116.

Starting point

Theorem (G.A. Freiman)

Let S be a finite set of integers with $k \geq 3$ elements and suppose that $|2S| \leq 2k - 1 + b$, where $0 \leq b \leq k - 3$. Then S is contained in an arithmetic progression of length $k + b$.

In particular

$3k - 4$ Theorem (G.A. Freiman)

Let S be a finite set of integers with $k \geq 3$ elements and suppose that $|2S| \leq 3k - 4$. Then there exist integers a and q such that $q > 0$ and

$$S \subseteq \{a, a + q, a + 2q, \dots, a + (|2X| - k)q\} .$$

Starting point

Theorem (G.A. Freiman)

Let S be a finite set of integers with $k \geq 3$ elements and suppose that $|2S| \leq 2k - 1 + b$, where $0 \leq b \leq k - 3$. Then S is contained in an arithmetic progression of length $k + b$.

In particular

$3k - 4$ Theorem (G.A. Freiman)

Let S be a finite set of integers with $k \geq 3$ elements and suppose that $|2S| \leq 3k - 4$. Then there exist integers a and q such that $q > 0$ and

$$S \subseteq \{a, a+q, a+2q, \ldots, a+(|2X|-k)q\} .$$

Starting point

Freiman studied also the case $|2S| \leq 3|S| - 3$ and $|2S| \leq 3|S| - 2$.

Theorem (G.A. Freiman)

Let S be a finite set of integers with $k \geq 2$ elements and suppose that

$$|2S| \leq 3k - 3.$$

Then one of the following holds:

(i) S is a subset of an arithmetic progression of size at most $2k - 1$;

(ii) S is a bi-arithmetic progression

$$S = \{a, a+d, \cdots, a+(i-1)d\} \cup \{b, b+d, \cdots, b+(j-1)d\}, i+j = k;$$

(iii) $k = 6$ and S has a determined structure.

Starting point

Freiman studied also the case $|2S| \leq 3|S| - 3$ and $|2S| \leq 3|S| - 2$.

Theorem (G.A. Freiman)

Let S be a finite set of integers with $k \geq 2$ elements and suppose that

$$|2S| \leq 3k - 3.$$

Then one of the following holds:

(i) S is a subset of an arithmetic progression of size at most $2k - 1$;

(ii) S is a bi-arithmetic progression

$$S = \{a, a + d, \cdots, a + (i - 1)d\} \cup \{b, b + d, \cdots, b + (j - 1)d\}, i + j = k;$$

(iii) $k = 6$ and S has a determined structure.

Basic definition

Definition

If S is a subset of a group $(G, \cdot)$, write

$$S^2 = SS := \{xy \mid x, y \in S\}.$$

S^2 is also called the square of S.

If G is an additive group, then we put

$$2S = S + S := \{x + y \mid x, y \in S\}.$$

$2S$ is also called the double of S.

Basic definition

Definition

If S is a subset of a group $(G, \cdot)$, write

$$S^2 = SS := \{xy \mid x, y \in S\}.$$

S^2 is also called the square of S.

If G is an additive group, then we put

$$2S = S + S := \{x + y \mid x, y \in S\}.$$

$2S$ is also called the double of S.

Main problems

Problem

Given S, find information about $|S^2|$.

Direct problems

Problem

Given some bound for $|S^2|$,

find information about the structure of S.

Inverse problems

Small doubling problems

By now, Freiman's theory had been extended tremendously, in many different directions.

It was shown by Freiman and others that problems in various fields may be looked at and treated as Structure Theory problems, including Additive and Combinatorial Number Theory, Group Theory, Integer Programming and Coding Theory.

J. Cilleuelo, **M. Silva**, **C. Vinuesa**, **H. Halberstam**, **N. Gill**, **B.J. Green**, **H. Helfgott**, **R. Jin**, **V.F. Lev**, **P.Y. Smeliansky** , **I.Z. Ruzsa**, **T. Sanders**, **T.C. Tao**, ...

Now let G be a **torsion-free** group.

Doubling problems

Let G be a group and S a finite subset of G.

Let α, β be real numbers.

Inverse problems of doubling type

What is the structure of S if $|S^2|$ satisfies

$$|S^2| \leq \alpha|S| + \beta?$$

The coefficient α, or more precisely the ratio $\frac{|S^2|}{|S|}$, is called the **doubling coefficient of** S.

Doubling problems

There are two main types of questions one may ask.

Question 1

What is the general type of structure that S can have if

$$|S^2| \leq \alpha|S| + \beta?$$

How behaves this type of structure when α increases?

Question 2

For a given (in general quite small) range of values for α find the precise (and possibly complete) description of those finite sets S which satisfy

$$|S^2| \leq \alpha|S| + \beta,$$

with α and $|\beta|$ **small**.

Doubling problems

There are two main types of questions one may ask.

Question 1

What is the general type of structure that S can have if

$$|S^2| \leq \alpha|S| + \beta?$$

How behaves this type of structure when α increases?

Question 2

For a given (in general quite small) range of values for α find the precise (and possibly complete) description of those finite sets S which satisfy

$$|S^2| \leq \alpha|S| + \beta,$$

with α and $|\beta|$ **small**.

Doubling problems

Question 1

What is the general type of structure that S can have if

$$|S^2| \leq \alpha|S| + \beta?$$

How behaves this type of structure when α increases?

Studied recently by many authors:

E. Breuillard, B. Green, I.Z. Ruzsa, T. Tao, . . .

Very powerful general results have been obtained (leading to a qualitatively complete structure theorem thanks to the concepts of nilprogressions and approximate groups).

But these results are not very precise quantitatively.

Small doubling problems

Question 2

For a given (in general quite small) range of values for α find the precise (and possibly complete) description of those finite sets S which satisfy

$$|S^2| \leq \alpha|S| + \beta,$$

with α and $|\beta|$ **small**.

Background - direct results - doubling coefficient 2

Proposition

If S is a non-empty finite subset of the group of the integers, then we have $$|2S| \geq 2|S| - 1.$$

More generally:

Theorem (J.H.B. Kemperman, *Indag. Mat.*, 1956)

If S is a non-empty finite subset of a torsion-free group, then we have $$|S^2| \geq 2|S| - 1.$$

Question

Is this bound sharp in any torsion-free group?

Background - direct results - doubling coefficient 2

Proposition

If S is a non-empty finite subset of the group of the integers, then we have $$|2S| \geq 2|S| - 1.$$

More generally:

Theorem (J.H.B. Kemperman, *Indag. Mat.*, 1956)

If S is a non-empty finite subset of a torsion-free group, then we have $$|S^2| \geq 2|S| - 1.$$

Question

Is this bound sharp in any torsion-free group?

Background - direct results - doubling coefficient 2

Proposition

If S is a non-empty finite subset of the group of the integers, then we have $$|2S| \geq 2|S| - 1.$$

More generally:

Theorem (J.H.B. Kemperman, *Indag. Mat.*, 1956)

If S is a non-empty finite subset of a torsion-free group, then we have $$|S^2| \geq 2|S| - 1.$$

Question

Is this bound sharp in any torsion-free group?

Progressions

Definition

If $a, r \neq 1$ are elements of a multiplicative group G, a geometric left (rigth) progression with ratio r and length n is the subset of G

$$\{a, ar, ar^2, \cdots, ar^{n-1}\} \ (\{a, ra, r^2a, \cdots, r^{n-1}a\}).$$

If G is an additive abelian group,

$$\{a, a+r, a+2r, \cdots, a+(n-1)r\}$$

is called an arithmetic progression with difference r and length n.

Definition

If $a, r \neq 1$ are elements of a multiplicative group G, a geometric left (rigth) progression with ratio r and length n is the subset of G

$$\{a, ar, ar^2, \cdots, ar^{n-1}\} \ (\{a, ra, r^2a, \cdots, r^{n-1}a\}).$$

If G is an additive abelian group,

$$\{a, a+r, a+2r, \cdots, a+(n-1)r\}$$

is called an arithmetic progression with difference r and length n.

An example - doubling coefficient 2

Example

If $S = \{a, ar, ar^2, \cdots, ar^{n-1}\}$ is a geometric progression in a torsion-free group and $ar = ra$, then

$$S^2 = \{a^2, a^2r, a^2r^2, \cdots, a^2r^{2n-2}\} \text{ has order } 2|S| - 1.$$

Background - inverse results - doubling coefficient 2

Theorem (G.A. Freiman, B.M. Schein, *Proc. Amer. Math. Soc.*, 1991)

If S is a finite subset of a torsion-free group, $|S| = k \geq 2$,

$$|S^2| = 2|S| - 1$$

if and only if

$S = \{a, aq, \cdots, aq^{k-1}\}$, and either $aq = qa$ or $aqa^{-1} = q^{-1}$.

In particular, if $|S^2| = 2|S| - 1$, then S is contained in a left coset of a cyclic subgroup of G.

Theorem (Y.O. Hamidoune, A.S. Lladó, O. Serra, *Combinatorica*, 1998)

If S is a finite subset of a torsion-free group G, $|S| = k \geq 4$, such that

$$|S^2| \leq 2|S|,$$

then there exist $a, q \in G$ such that

$$S = \{a, aq, \cdots, aq^k\} \setminus \{c\}, \text{ with } c \in \{a, aq\}.$$

Small doubling problems with doubling coefficient 3

Theorem (G.A. Freiman, 1959)

Let S be a finite set of integers with $k \geq 3$ elements and suppose that

$$|2S| \leq 3k - 4.$$

Then S is contained in an arithmetic progression of size $2k - 3$.

Conjecture (G.A. Freiman)

If G is any torsion-free group, S a finite subset of G, $|S| \geq 4$, and

$$|S^2| \leq 3|S| - 4,$$

then S is contained in a geometric progression of length at most $2|S| - 3$.

Small doubling problems with doubling coefficient 3

Theorem (G.A. Freiman, 1959)

Let S be a finite set of integers with $k \geq 3$ elements and suppose that

$$|2S| \leq 3k - 4.$$

Then S is contained in an arithmetic progression of size $2k - 3$.

Conjecture (G.A. Freiman)

If G is any torsion-free group, S a finite subset of G, $|S| \geq 4$, and

$$|S^2| \leq 3|S| - 4,$$

then S is contained in a geometric progression of length at most $2|S| - 3$.

Theorem (G.A. Freiman)

Let S be a finite set of integers with $k \geq 2$ elements and suppose that

$$|2S| \leq 3k - 3.$$

Then one of the following holds:

(i) S is contained in an arithmetic progression of size at most $2k-1$;

(ii) S is a bi-arithmetic progression

$S = \{a, a+q, a+2q, \cdots, a+(i-1)q\} \cup \{b, b+q, a+2q, \cdots, b+(j-1)q\}$;

(iii) $k = 6$ and S has a determined structure.

Problem

Let G be any torsion-free group, S a finite subset of G,

$$|S| \geq 3.$$

What is the structure of S if

$$|S^2| \leq 3|S| - 3?$$

Small doubling problems with doubling coefficient 3

Freiman studied also the case $|2S| = 3|S| - 2$, S a finite set of integers. He proved that, with the exception of some cases with $|S|$ small, then either S is contained in an arithmetic progression or it is the union of two arithmetic progressions with same difference.

Conjecture (G.A. Freiman)

If G is any torsion-free group, S a finite subset of G, $|S| \geq 11$, and $|S^2| \leq 3|S| - 2$,

then S is contained in a geometric progression of length at most $2|S| + 1$ or

it is the union of two geometric progressions with same ratio.

Small doubling problems have been studied

in abelian groups

by many authors:

Y.O. Hamidoune, B. Green, M. Kneser,

A.S. Lladó, A. Plagne, P.P. Palfy,

I.Z. Ruzsa, O. Serra, Y.V. Stanchescu, . . .

New results

In a series of papers with

Gregory Freiman, Marcel Herzog, Mercede Maj,

Yonutz Stanchescu, Alain Plagne, Derek Robinson

we studied Freiman's conjectures

and more generally small doubling problems

in the class of orderable groups.

New results

New results

New results

New results

New results

New results

Papers

G.A. Freiman, M. Herzog, P. L., M. Maj

Small doubling in ordered groups

J. Australian Math. Soc., **96** (2014), no. 3, 316-325.

Papers

G.A. Freiman, M. Herzog, P.L., M. Maj, Y.V. Stanchescu

$$\textit{Direct and inverse problems in additive number theory and in non}-\textit{abelian group theory}$$

European J. Combin. **40** (2014), 42-54.

$$\textit{A small doubling structure theorem in a Baumslag}-\textit{Solitar group}$$

European J. Combin. **44** (2015), 106-124.

Papers

G.A. Freiman, M. Herzog, P. L., M. Maj, A. Plagne, D.J.S. Robinson, Y.V. Stanchescu

On the structure of subsets of an orderable group, with some small doubling properties

J. Algebra, **445** (2016), 307-326.

G.A. Freiman, M. Herzog, P. L., M. Maj, A. Plagne, Y.V. Stanchescu

Small doubling in ordered groups : generators and structures,

Groups Geom. Dyn., **11** (2017), 585-612.

Papers

G.A. Freiman, M. Herzog, P. L., M. Maj, Y.V. Stanchescu

Small doubling in ordered nilpotent group of class 2,

European Journal of Combinatorics, (2017)

http://dx.doi.org/10.1016/j.ejc.2017.07.006,

to appear.

Ordered groups

Definition

Let G be a group and suppose that a total order relation $\leq$ is defined on the set G.
We say that $(G, \leq)$ is an *ordered group* if for all $a, b, x, y \in G$, the inequality $a \leq b$ implies that $xay \leq xby$.

Definition

A group G is *orderable* if there exists a total order relation $\leq$ on the set G, such that $(G, \leq)$ is an ordered group.

Definition

Let G be a group and suppose that a total order relation $\leq$ is defined on the set G.
We say that $(G, \leq)$ is an *ordered group* if for all $a, b, x, y \in G$, the inequality $a \leq b$ implies that $xay \leq xby$.

Definition

A group G is *orderable* if there exists a total order relation $\leq$ on the set G, such that $(G, \leq)$ is an ordered group.

Orderable groups

The following properties of ordered groups follow easily from the definition.

- If $a < 1$, then $a^{-1} > 1$ and conversely.
- If $a > 1$, then $x^{-1}ax > 1$.
- If $a > b$ and n is a positive integer, then $a^n > b^n$ and $a^{-n} < b^{-n}$.
- G is torsion-free.

Lemma (B.H. Neumann)

Let $(G, <)$ be an ordered group and let $a, b \in G$.
If $a^n b = ba^n$ for some integer $n \neq 0$, then $ab = ba$.

Orderable groups

The following properties of ordered groups follow easily from the definition.

- If $a < 1$, then $a^{-1} > 1$ and conversely.
- If $a > 1$, then $x^{-1}ax > 1$.
- If $a > b$ and n is a positive integer, then $a^n > b^n$ and $a^{-n} < b^{-n}$.
- G is torsion-free.

Lemma (B.H. Neumann)

Let $(G, <)$ be an ordered group and let $a, b \in G$.
If $a^n b = ba^n$ for some integer $n \neq 0$, then $ab = ba$.

Orderable groups

Theorem (F.W. Levi)

An **abelian group** G is orderable if and only if it is torsion-free.

Theorem (K. Iwasawa - A.I. Mal'cev - B.H. Neumann)

The class of orderable groups contains the class of **torsion-free nilpotent groups**.

Free groups are orderable.

The group

$$\langle x, c \mid x^{-1}cx = c^{-1} \rangle$$

is not an orderable group.

Orderable groups

Theorem (F.W. Levi)

An **abelian group** G is orderable if and only if it is torsion-free.

Theorem (K. Iwasawa - A.I. Mal'cev - B.H. Neumann)

The class of orderable groups contains the class of **torsion-free nilpotent groups**.

Free groups are orderable.

The group

$$\langle x, c \mid x^{-1}cx = c^{-1} \rangle$$

is not an orderable group.

Orderable groups

Theorem (F.W. Levi)

An **abelian group** G is orderable if and only if it is torsion-free.

Theorem (K. Iwasawa - A.I. Mal'cev - B.H. Neumann)

The class of orderable groups contains the class of **torsion-free nilpotent groups**.

Free groups are orderable.

The group

$$\langle x, c \mid x^{-1}cx = c^{-1} \rangle$$

is not an orderable group.

Orderable groups

Theorem (F.W. Levi)

An **abelian group** G is orderable if and only if it is torsion-free.

Theorem (K. Iwasawa - A.I. Mal'cev - B.H. Neumann)

The class of orderable groups contains the class of **torsion-free nilpotent groups**.

Free groups are orderable.

The group

$$\langle x, c \mid x^{-1}cx = c^{-1} \rangle$$

is not an orderable group.

More information concerning orderable groups may be found, for example, in

R. Botto Mura and A. Rhemtulla, *Orderable groups*, Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, Inc., New York and Basel, 1977.

A.M.W. Glass, *Partially ordered groups*, World Scientific Publishing Co., Series in Algebra, **v. 7**, 1999.

Orderable groups

Any orderable group is an R-group.

A group G is an **R-group** if, with $a, b \in G$,

$$a^n = b^n, \ \ n \neq 0, \ \text{implies } a = b.$$

Any orderable group is an R*-group.

A group G is an **R*-group** if, with $a, b, g_1, \cdots, g_n \in G$,

$$a^{g_1} \cdots a^{g_n} = b^{g_1} \cdots b^{g_n} \ \text{implies } a = b.$$

A metabelian R*-group is orderable.

Orderable groups

Any orderable group is an R-group.

A group G is an **R-group** if, with $a, b \in G$,

$$a^n = b^n, \ n \neq 0, \ \text{implies } a = b.$$

Any orderable group is an R*-group.

A group G is an **R*-group** if, with $a, b, g_1, \cdots, g_n \in G$,

$$a^{g_1} \cdots a^{g_n} = b^{g_1} \cdots b^{g_n} \ \text{implies } a = b.$$

A metabelian R*-group is orderable.

Orderable groups

Any orderable group is an R-group.

A group G is an **R-group** if, with $a, b \in G$,

$$a^n = b^n, \ n \neq 0, \ \text{implies } a = b.$$

Any orderable group is an R*-group.

A group G is an **R*-group** if, with $a, b, g_1, \cdots, g_n \in G$,

$$a^{g_1} \cdots a^{g_n} = b^{g_1} \cdots b^{g_n} \text{ implies } a = b.$$

A metabelian R*-group is orderable.

Small doubling in orderable groups: $|S^2| \leq 3|S| - 4$

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, *J. Austral. Math. Soc.*, 2014)

Let $(G, \leq)$ be an ordered group and let S be a finite subset of G of size $k \geq 3$.
Assume that

$$t = |S^2| \leq 3|S| - 4.$$

Then $\langle S \rangle$ is abelian. Moreover, there exist $a, q \in G$, such that $qa = aq$ and S is a subset of

$$\{a, aq, aq^2, \cdots, aq^{t-k}\}.$$

Small doubling in orderable groups: $|S^2| \leq 3|S| - 4$

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, *J. Austral. Math. Soc.*, 2014)

Let $(G, \leq)$ be an ordered group and let S be a finite subset of G of size $k \geq 3$.
Assume that

$$t = |S^2| \leq 3|S| - 4.$$

Then $\langle S \rangle$ is abelian. Moreover, there exist $a, q \in G$, such that $qa = aq$ and S is a subset of

$$\{a, aq, aq^2, \cdots, aq^{t-k}\}.$$

Small doubling in orderable groups: $|S^2| \leq 3|S| - 4$

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, *J. Austral. Math. Soc.*, 2014)

Let $(G, \leq)$ be an ordered group and let S be a finite subset of G of size $k \geq 3$.
Assume that

$$t = |S^2| \leq 3|S| - 4.$$

Then $\langle S \rangle$ is abelian. Moreover, there exist $a, q \in G$, such that $qa = aq$ and S is a subset of

$$\{a, aq, aq^2, \cdots, aq^{t-k}\}.$$

Small doubling in orderable groups: $|S^2| \leq 3|S| - 3$

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, *J. Austral. Math. Soc.*, 2014)

Let $(G, \leq)$ be an ordered group and let S be a finite subset of G, $|S| \geq 3$. Assume that

$$|S^2| \leq 3|S| - 3.$$

Then $\langle S \rangle$ is abelian.

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, *J. Austral. Math. Soc.*, 2014)

Let $(G, \leq)$ be an ordered group and let S be a finite subset of G, $|S| \geq 3$. Assume that

$$|S^2| \leq 3|S| - 3.$$

Then $\langle S \rangle$ is abelian.

Small doubling in orderable groups: $|S^2| \leq 3|S| - 3$

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, A. Plagne, Y.V. Stanchescu, *Groups Geom. Dyn.*, 2017)

Let G be an ordered group and let S be a finite subset of G, $|S| \geq 3$. If

$$|S^2| \leq 3|S| - 3,$$

then $\langle S \rangle$ is abelian, at most 3-generated and one of the following holds:

(1) $|S| \leq 10$;

(2) S is a subset of a geometric progression of length at most $2|S| - 1$;

(3) $S = \{ac^t \mid 0 \leq t \leq t_1 - 1\} \cup \{bc^t \mid 0 \leq t \leq t_2 - 1\}$.

Small doubling in orderable groups: $|S^2| \leq 3|S| - 3$

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, A. Plagne, Y.V. Stanchescu, *Groups Geom. Dyn.*, 2017)

Let G be an ordered group and let S be a finite subset of G, $|S| \geq 3$. If

$$|S^2| \leq 3|S| - 3,$$

then $\langle S \rangle$ is abelian, at most 3-generated and one of the following holds:

(1) $|S| \leq 10$;

(2) S is a subset of a geometric progression of length at most $2|S| - 1$;

(3) $S = \{ac^t \mid 0 \leq t \leq t_1 - 1\} \cup \{bc^t \mid 0 \leq t \leq t_2 - 1\}$.

Small doubling in orderable groups: $|S^2| \leq 3|S| - 3$

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, A. Plagne, Y.V. Stanchescu, *Groups Geom. Dyn.*, 2017)

Let G be an ordered group and let S be a finite subset of G, $|S| \geq 3$. If

$$|S^2| \leq 3|S| - 3,$$

then $\langle S \rangle$ is abelian, at most 3-generated and one of the following holds:

(1) $|S| \leq 10$;

(2) S is a subset of a geometric progression of length at most $2|S| - 1$;

(3) $S = \{ac^t \mid 0 \leq t \leq t_1 - 1\} \cup \{bc^t \mid 0 \leq t \leq t_2 - 1\}$.

Small doubling in orderable groups: $|S^2| \leq 3|S| - 3$

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, A. Plagne, Y.V. Stanchescu, *Groups Geom. Dyn.*, 2017)

Let G be an ordered group and let S be a finite subset of G, $|S| \geq 3$. If

$$|S^2| \leq 3|S| - 3,$$

then $\langle S \rangle$ is abelian, at most 3-generated and one of the following holds:

(1) $|S| \leq 10$;

(2) S is a subset of a geometric progression of length at most $2|S| - 1$;

(3) $S = \{ac^t \mid 0 \leq t \leq t_1 - 1\} \cup \{bc^t \mid 0 \leq t \leq t_2 - 1\}$.

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, A. Plagne, Y.V. Stanchescu, *Groups Geom. Dyn.*, 2017)

Let G be an ordered group and let S be a finite subset of G, $|S| \geq 3$. If

$$|S^2| \leq 3|S| - 3,$$

then $\langle S \rangle$ is abelian, at most 3-generated and one of the following holds:

(1) $|S| \leq 10$;

(2) S is a subset of a geometric progression of length at most $2|S| - 1$;

(3) $S = \{ac^t \mid 0 \leq t \leq t_1 - 1\} \cup \{bc^t \mid 0 \leq t \leq t_2 - 1\}$.

Small doubling in orderable groups: $\langle S \rangle$ abelian?

Questions

What about $\langle S \rangle$ if S is a subset of an orderable group and

$$|S^2| \leq 3|S| - 2?$$

Is it abelian? Is it abelian if $|S|$ is big enough?

Small doubling in orderable groups: $\langle S \rangle$ abelian?

Remark

There exists an ordered group G with a subset S of order k (for any k) such that $\langle S \rangle$ is not abelian and $|S^2| = 3k - 2$.

Example

Let

$$G = \langle a, b \mid a^b = a^2 \rangle,$$

the Baumslag-Solitar group $\mathcal{BS}(1,2)$ and

$$S = \{b, ba, ba^2, \cdots, ba^{k-1}\}.$$

Then

$$S^2 = \{b^2, b^2a, b^2a^2, b^2a^3, \cdots, b^2a^{3k-3}\}.$$

Thus $\langle S \rangle$ is non-abelian and $|S^2| = 3k - 2$.

Small doubling in orderable groups: $\langle S \rangle$ abelian?

Remark

There exists an ordered group G with a subset S of order k (for any k) such that $\langle S \rangle$ is not abelian and $|S^2| = 3k - 2$.

Example

Let

$$G = \langle a, b \mid a^b = a^2 \rangle,$$

the Baumslag-Solitar group $\mathcal{BS}(1,2)$ and

$$S = \{b, ba, ba^2, \cdots, ba^{k-1}\}.$$

Then

$$S^2 = \{b^2, b^2a, b^2a^2, b^2a^3, \cdots, b^2a^{3k-3}\}.$$

Thus $\langle S \rangle$ is non-abelian and $|S^2| = 3k - 2$.

The structure of $\langle S \rangle$ if $|S^2| = 3|S| - 2$

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, A. Plagne, Y.V. Stanchescu, *Groups Geom. Dyn.*, 2017)

Let G be an ordered group and let S be a finite subset of G, $|S| \geq 4$. If

$$|S^2| = 3|S| - 2$$

then one of the following holds:

(1) $\langle S \rangle$ is an abelian group, at most 4-generated;

(2) $\langle S \rangle = \langle a, b \mid [a, b] = c, [c, a] = [c, b] = 1 \rangle$. In particular $\langle S \rangle$ is a nilpotent group of class 2;

(3) $\langle S \rangle = \langle a, b \mid a^b = a^2 \rangle$. Therefore $\langle S \rangle$ is the *Baumslag-Solitar group* $\mathcal{BS}(1, 2)$;

(4) $\langle S \rangle = \langle a \rangle \times \langle b, c \mid c^b = c^2 \rangle$;

(5) $\langle S \rangle = \langle a, b \mid a^{b^2} = aa^b, aa^b = a^b a \rangle$.

The structure of $\langle S\rangle$ if $|S^2| = 3|S| - 2$

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, A. Plagne, Y.V. Stanchescu, *Groups Geom. Dyn.*, 2017)

Let G be an ordered group and let S be a finite subset of G, $|S| \geq 4$. If

$$|S^2| = 3|S| - 2$$

then one of the following holds:

(1) $\langle S\rangle$ is an abelian group, at most 4-generated;

(2) $\langle S\rangle = \langle a, b \mid [a, b] = c, [c, a] = [c, b] = 1\rangle$. In particular $\langle S\rangle$ is a nilpotent group of class 2;

(3) $\langle S\rangle = \langle a, b \mid a^b = a^2\rangle$. Therefore $\langle S\rangle$ is the *Baumslag-Solitar group* $\mathcal{BS}(1, 2)$;

(4) $\langle S\rangle = \langle a\rangle \times \langle b, c \mid c^b = c^2\rangle$;

(5) $\langle S\rangle = \langle a, b \mid a^{b^2} = aa^b, aa^b = a^ba\rangle$.

The structure of $\langle S \rangle$ if $|S^2| = 3|S| - 2$

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, A. Plagne, Y.V. Stanchescu, *Groups Geom. Dyn.*, 2017)

Let G be an ordered group and let S be a finite subset of G, $|S| \geq 4$. If

$$|S^2| = 3|S| - 2$$

then one of the following holds:

(1) $\langle S \rangle$ is an abelian group, at most 4-generated;

(2) $\langle S \rangle = \langle a, b \mid [a, b] = c, [c, a] = [c, b] = 1 \rangle$. In particular $\langle S \rangle$ is a nilpotent group of class 2;

(3) $\langle S \rangle = \langle a, b \mid a^b = a^2 \rangle$. Therefore $\langle S \rangle$ is the *Baumslag-Solitar group* $\mathcal{BS}(1, 2)$;

(4) $\langle S \rangle = \langle a \rangle \times \langle b, c \mid c^b = c^2 \rangle$;

(5) $\langle S \rangle = \langle a, b \mid a^{b^2} = aa^b, aa^b = a^b a \rangle$.

The structure of $\langle S\rangle$ if $|S^2| = 3|S| - 2$

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, A. Plagne, Y.V. Stanchescu, *Groups Geom. Dyn.*, 2017)

Let G be an ordered group and let S be a finite subset of G, $|S| \geq 4$. If

$$|S^2| = 3|S| - 2$$

then one of the following holds:

(1) $\langle S\rangle$ is an abelian group, at most 4-generated;

(2) $\langle S\rangle = \langle a, b \mid [a, b] = c, [c, a] = [c, b] = 1\rangle$. In particular $\langle S\rangle$ is a nilpotent group of class 2;

(3) $\langle S\rangle = \langle a, b \mid a^b = a^2\rangle$. Therefore $\langle S\rangle$ is the *Baumslag-Solitar group* $\mathcal{BS}(1, 2)$;

(4) $\langle S\rangle = \langle a\rangle \times \langle b, c \mid c^b = c^2\rangle$;

(5) $\langle S\rangle = \langle a, b \mid a^{b^2} = aa^b, aa^b = a^b a\rangle$.

The structure of $\langle S\rangle$ if $|S^2| = 3|S| - 2$

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, A. Plagne, Y.V. Stanchescu, *Groups Geom. Dyn.*, 2017)

Let G be an ordered group and let S be a finite subset of G, $|S| \geq 4$. If

$$|S^2| = 3|S| - 2$$

then one of the following holds:

(1) $\langle S\rangle$ is an abelian group, at most 4-generated;

(2) $\langle S\rangle = \langle a, b \mid [a,b] = c, [c,a] = [c,b] = 1\rangle$. In particular $\langle S\rangle$ is a nilpotent group of class 2;

(3) $\langle S\rangle = \langle a, b \mid a^b = a^2\rangle$. Therefore $\langle S\rangle$ is the *Baumslag-Solitar group* $\mathcal{BS}(1,2)$;

(4) $\langle S\rangle = \langle a\rangle \times \langle b, c \mid c^b = c^2\rangle$;

(5) $\langle S\rangle = \langle a, b \mid a^{b^2} = aa^b, aa^b = a^b a\rangle$.

The structure of $\langle S\rangle$ if $|S^2| = 3|S| - 2$

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, A. Plagne, Y.V. Stanchescu, *Groups Geom. Dyn.*, 2017)

Let G be an ordered group and let S be a finite subset of G, $|S| \geq 4$. If
$$|S^2| = 3|S| - 2$$
then one of the following holds:

(1) $\langle S\rangle$ is an abelian group, at most 4-generated;

(2) $\langle S\rangle = \langle a, b \,|[a, b] = c, [c, a] = [c, b] = 1\rangle$. In particular $\langle S\rangle$ is a nilpotent group of class 2;

(3) $\langle S\rangle = \langle a, b \mid a^b = a^2\rangle$. Therefore $\langle S\rangle$ is the *Baumslag-Solitar group* $\mathcal{BS}(1, 2)$;

(4) $\langle S\rangle = \langle a\rangle \times \langle b, c \mid c^b = c^2\rangle$;

(5) $\langle S\rangle = \langle a, b \mid a^{b^2} = aa^b, aa^b = a^b a\rangle$.

The structure of S if $|S^2| = 3|S| - 2$: $\langle S \rangle$ abelian

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, A. Plagne, Y.V. Stanchescu, *Groups Geom. Dyn.*, 2017)

Let G be an ordered group and let S be a subset of G of finite size $k > 2$. If

$$|S^2| = 3k - 2,$$

and $\langle S \rangle$ is **abelian**, then one of the following possibilities occurs:

(1) $|S| \leq 11$;

(2) S is a subset of a geometric progression of length at most $2|S| + 1$;

(3) S is contained in the union of two geometric progressions with the same ratio.

The structure of S if $|S^2| = 3|S| - 2$: $\langle S \rangle$ abelian

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, A. Plagne, Y.V. Stanchescu, *Groups Geom. Dyn.*, 2017)

Let G be an ordered group and let S be a subset of G of finite size $k > 2$. If

$$|S^2| = 3k - 2,$$

and $\langle S \rangle$ is **abelian**, then one of the following possibilities occurs:

(1) $|S| \leq 11$;

(2) S is a subset of a geometric progression of length at most $2|S| + 1$;

(3) S is contained in the union of two geometric progressions with the same ratio.

The structure of S if $|S^2| = 3|S| - 2$: $\langle S \rangle$ abelian

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, A. Plagne, Y.V. Stanchescu, *Groups Geom. Dyn.*, 2017)

Let G be an ordered group and let S be a subset of G of finite size $k > 2$. If

$$|S^2| = 3k - 2,$$

and $\langle S \rangle$ is **abelian**, then one of the following possibilities occurs:

(1) $|S| \leq 11$;

(2) S is a subset of a geometric progression of length at most $2|S| + 1$;

(3) S is contained in the union of two geometric progressions with the same ratio.

The structure of S if $|S^2| = 3|S| - 2$: $\langle S \rangle$ abelian

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, A. Plagne, Y.V. Stanchescu, *Groups Geom. Dyn.*, 2017)

Let G be an ordered group and let S be a subset of G of finite size $k > 2$. If

$$|S^2| = 3k - 2,$$

and $\langle S \rangle$ is **abelian**, then one of the following possibilities occurs:

(1) $|S| \leq 11$;

(2) S is a subset of a geometric progression of length at most $2|S| + 1$;

(3) S is contained in the union of two geometric progressions with the same ratio.

The structure of S if $|S^2| = 3|S| - 2$: $\langle S \rangle$ abelian

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, A. Plagne, Y.V. Stanchescu, *Groups Geom. Dyn.*, 2017)

Let G be an ordered group and let S be a subset of G of finite size $k > 2$. If

$$|S^2| = 3k - 2,$$

and $\langle S \rangle$ is **abelian**, then one of the following possibilities occurs:

(1) $|S| \leq 11$;

(2) S is a subset of a geometric progression of length at most $2|S| + 1$;

(3) S is contained in the union of two geometric progressions with the same ratio.

The structure of S if $|S^2| = 3|S| - 2$: $\langle S \rangle$ nilpotent n.ab.

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, Y.V. Stanchescu, *European J. Combin.*, 2017)

Let G be a torsion-free **nilpotent** group and let S be a subset of G of size $k \geq 4$ with $\langle S \rangle$ **non-abelian**.
Then $|S^2| = 3k - 2$ if and only if there exist $a, b, c \in G$ and non-negative integers i, j such that

$$S = \{a, ac, \cdots, ac^i, b, bc, \cdots, bc^j\},$$

with $1 + i + 1 + j = k$, $c \neq 1$ and $[a, b] = c^{\pm 1}$.

The structure of S if $|S^2| = 3|S| - 2$: $\langle S \rangle$ nilpotent n.ab.

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, Y.V. Stanchescu, *European J. Combin.*, 2017)

Let G be a torsion-free **nilpotent** group and let S be a subset of G of size $k \geq 4$ with $\langle S \rangle$ **non-abelian**.
Then $|S^2| = 3k - 2$ if and only if there exist $a, b, c \in G$ and non-negative integers i, j such that

$$S = \{a, ac, \cdots, ac^i, b, bc, \cdots, bc^j\},$$

with $1 + i + 1 + j = k$, $c \neq 1$ and $[a, b] = c^{\pm 1}$.

The structure of S if $|S^2| = 3|S| - 2$: $\langle S \rangle$ non-abelian

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, A. Plagne, D.J.S. Robinson, Y.V. Stanchescu, *J. Algebra*, 2016)

Let G be an ordered group and let S be a subset of G of finite size $k > 2$. If
$$|S^2| = 3k - 2,$$
and $\langle S \rangle$ is **non-abelian**, then one of the following statements holds:

(1) $|S| \leq 4$;

(2) $S = \{x, xc, xc^2, \cdots, xc^{k-1}\}$, where $c^x = c^2$ or $(c^2)^x = c$;

(3) $S = \{a, ac, ac^2, \cdots, ac^i, b, bc, bc^2, \cdots, bc^j\}$, with $1 + i + 1 + j = k$ and $ab = bac$ *or* $ba = abc$, $ac = ca$, $bc = cb$, $c > 1$.

Conversely if S has the structure in (2) and (3), then $|S^2| = 3|S| - 2$.

The structure of S if $|S^2| = 3|S| - 2$: $\langle S \rangle$ non-abelian

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, A. Plagne, D.J.S. Robinson, Y.V. Stanchescu, *J. Algebra*, 2016)

Let G be an ordered group and let S be a subset of G of finite size $k > 2$. If
$$|S^2| = 3k - 2,$$
and $\langle S \rangle$ is **non-abelian**, then one of the following statements holds:

(1) $|S| \leq 4$;

(2) $S = \{x, xc, xc^2, \cdots, xc^{k-1}\}$, where $c^x = c^2$ or $(c^2)^x = c$;

(3) $S = \{a, ac, ac^2, \cdots, ac^i, b, bc, bc^2, \cdots, bc^j\}$, with $1 + i + 1 + j = k$ and $ab = bac$ *or* $ba = abc$, $ac = ca$, $bc = cb$, $c > 1$.

Conversely if S has the structure in (2) and (3), then $|S^2| = 3|S| - 2$.

The structure of S if $|S^2| = 3|S| - 2$: $\langle S \rangle$ non-abelian

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, A. Plagne, D.J.S. Robinson, Y.V. Stanchescu, *J. Algebra*, 2016)

Let G be an ordered group and let S be a subset of G of finite size $k > 2$. If $$|S^2| = 3k - 2,$$
and $\langle S \rangle$ is **non-abelian**, then one of the following statements holds:

(1) $|S| \leq 4$;

(2) $S = \{x, xc, xc^2, \cdots, xc^{k-1}\}$, where $c^x = c^2$ or $(c^2)^x = c$;

(3) $S = \{a, ac, ac^2, \cdots, ac^i, b, bc, bc^2, \cdots, bc^j\}$, with $1 + i + 1 + j = k$ and $ab = bac$ or $ba = abc$, $ac = ca$, $bc = cb$, $c > 1$.

Conversely if S has the structure in (2) and (3), then $|S^2| = 3|S| - 2$.

The structure of S if $|S^2| = 3|S| - 2$: $\langle S \rangle$ non-abelian

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, A. Plagne, D.J.S. Robinson, Y.V. Stanchescu, *J. Algebra*, 2016)

Let G be an ordered group and let S be a subset of G of finite size $k > 2$. If
$$|S^2| = 3k - 2,$$
and $\langle S \rangle$ is **non-abelian**, then one of the following statements holds:

(1) $|S| \leq 4$;

(2) $S = \{x, xc, xc^2, \cdots, xc^{k-1}\}$, where $c^x = c^2$ or $(c^2)^x = c$;

(3) $S = \{a, ac, ac^2, \cdots, ac^i, b, bc, bc^2, \cdots, bc^j\}$, with $1 + i + 1 + j = k$ and $ab = bac$ or $ba = abc$, $ac = ca$, $bc = cb$, $c > 1$.

Conversely if S has the structure in (2) and (3), then $|S^2| = 3|S| - 2$.

The structure of S if $|S^2| = 3|S| - 2$: $\langle S \rangle$ non-abelian

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, A. Plagne, D.J.S. Robinson, Y.V. Stanchescu, *J. Algebra*, 2016)

Let G be an ordered group and let S be a subset of G of finite size $k > 2$. If

$$|S^2| = 3k - 2,$$

and $\langle S \rangle$ is **non-abelian**, then one of the following statements holds:

(1) $|S| \leq 4$;

(2) $S = \{x, xc, xc^2, \cdots, xc^{k-1}\}$, where $c^x = c^2$ or $(c^2)^x = c$;

(3) $S = \{a, ac, ac^2, \cdots, ac^i, b, bc, bc^2, \cdots, bc^j\}$, with $1 + i + 1 + j = k$ and $ab = bac$ *or* $ba = abc$, $ac = ca$, $bc = cb$, $c > 1$.

Conversely if S has the structure in (2) and (3), then $|S^2| = 3|S| - 2$.

The structure of S if $|S^2| = 3|S| - 2$: $\langle S \rangle$ non-abelian

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, A. Plagne, D.J.S. Robinson, Y.V. Stanchescu, *J. Algebra*, 2016)

Let G be an ordered group and let S be a subset of G of finite size $k > 2$. If
$$|S^2| = 3k - 2,$$
and $\langle S \rangle$ is **non-abelian**, then one of the following statements holds:

(1) $|S| \leq 4$;

(2) $S = \{x, xc, xc^2, \cdots, xc^{k-1}\}$, where $c^x = c^2$ or $(c^2)^x = c$;

(3) $S = \{a, ac, ac^2, \cdots, ac^i, b, bc, bc^2, \cdots, bc^j\}$, with $1 + i + 1 + j = k$ and $ab = bac$ *or* $ba = abc$, $ac = ca$, $bc = cb$, $c > 1$.

Conversely if S has the structure in (2) and (3), then $|S^2| = 3|S| - 2$.

Some methods - $\langle S \rangle$ abelian

In order to study the structure of S if $\langle S \rangle$ is abelian, we use some ideas suggested by Gregory Freiman.

Definition

Let A be a finite subset of an abelian group $(G, +)$ and B a finite subset of an abelian group $(H, +)$.
A map $\varphi : A \longrightarrow B$ is a Freiman isomorphism if it is bijective and from

$$a_1 + a_2 = b_1 + b_2$$

it follows

$$\varphi(a_1) + \varphi(a_2) = \varphi(b_1) + \varphi(b_2).$$

A is Freiman isomorphic to B if there exists a Freiman isomorphism $\varphi : A \longrightarrow B$.

Some methods - $\langle S \rangle$ abelian

In order to study the structure of S if $\langle S \rangle$ is abelian, we use some ideas suggested by Gregory Freiman.

Definition

Let A be a finite subset of an abelian group $(G, +)$ and B a finite subset of an abelian group $(H, +)$.
A map $\varphi : A \longrightarrow B$ is a Freiman isomorphism if it is bijective and from

$$a_1 + a_2 = b_1 + b_2$$

it follows

$$\varphi(a_1) + \varphi(a_2) = \varphi(b_1) + \varphi(b_2).$$

A is Freiman isomorphic to B if there exists a Freiman isomorphism $\varphi : A \longrightarrow B$.

Some methods - $\langle S \rangle$ abelian

In order to study the structure of S if $\langle S \rangle$ is abelian, we use some ideas suggested by Gregory Freiman.

Definition

Let A be a finite subset of an abelian group $(G,+)$ and B a finite subset of an abelian group $(H,+)$.
A map $\varphi : A \longrightarrow B$ is a Freiman isomorphism if it is bijective and from

$$a_1 + a_2 = b_1 + b_2$$

it follows

$$\varphi(a_1) + \varphi(a_2) = \varphi(b_1) + \varphi(b_2).$$

A is Freiman isomorphic to B if there exists a Freiman isomorphism $\varphi : A \longrightarrow B$.

Some methods - $\langle S \rangle$ abelian

Remark

If A and B are Freiman isomorphic, then

$$|A| = |B| \text{ and } |2A| = |2B|.$$

Remark

If $\varphi : A \longrightarrow B$ is a Freiman isomorphism and

$$A = \{a, a+d, a+2d, \cdots, a+(k-1)d\}$$

is an arithmetic progression with difference d, then B is an arithmetic progression with difference $\varphi(a+d) - \varphi(a)$.

Some methods - $\langle S \rangle$ abelian

Remark

If A and B are Freiman isomorphic, then

$$|A| = |B| \text{ and } |2A| = |2B|.$$

Remark

If $\varphi : A \longrightarrow B$ is a Freiman isomorphism and

$$A = \{a, a+d, a+2d, \cdots, a+(k-1)d\}$$

is an arithmetic progression with difference d, then B is an arithmetic progression with difference $\varphi(a+d) - \varphi(a)$.

Some methods - $\langle S\rangle = \mathcal{BS}(1,2)$ or $\langle S\rangle = \langle a\rangle \times \mathcal{BS}(1,2)$

In order to study the structure of S if

$$\langle S\rangle = \mathcal{BS}(1,2),$$

or

$$\langle S\rangle = \langle a\rangle \times \mathcal{BS}(1,2),$$

we can use dilates.

Dilates

Subsets of $\mathbb{Z}$ of the form

$$r * A := \{rx \mid x \in A\},$$

where r is a **positive** integer and A is a **finite** subset of $\mathbb{Z}$, are called *r-dilates*.

Sums of dilates are defined as usually:

$$r_1 * A + r_2 * A = \{r_1x_1 + r_2x_2 \mid x_1 \in A_1,\ x_2 \in A_2\}.$$

Dilates

Subsets of $\mathbb{Z}$ of the form

$$r * A := \{rx \mid x \in A\},$$

where r is a **positive** integer and A is a **finite** subset of $\mathbb{Z}$, are called *r-dilates*.

Sums of dilates are defined as usually:

$$r_1 * A + r_2 * A = \{r_1x_1 + r_2x_2 \mid x_1 \in A_1,\ x_2 \in A_2\}.$$

Dilates

These sums have been recently studied in different situations by **Bukh, Cilleruelo, Hamidoune, Plagne, Rué, Silva, Vinuesa** and others.

In particular, they examined sums of two dilates of the form

$$A + r * A = \{a + rb \mid \; a, b \in A\}$$

and solved various *direct* and *inverse* problems concerning their sizes.

The group $\mathcal{BS}(1,2) = \langle a, b \mid a^b = a^2 \rangle$

Theorem

Suppose that $S = b^r a^A \subseteq \mathcal{BS}(1,2)$, *where* $r \in \mathbb{Z}, r \geq 0$ *and* A *is a finite subset of* $\mathbb{Z}$. *Then*

$$S^2 = b^{2r} a^{2^r * A + A}$$

and

$$|S^2| = |2^r * A + A| = |A + 2^r * A|.$$

Some methods - $\langle S\rangle = \langle a, b \mid a^{b^2} = aa^b, aa^b = a^b a\rangle$

In order to study the structure of S if

$$\langle S\rangle = \langle a, b \mid a^{b^2} = aa^b, aa^b = a^b a\rangle,$$

we notice that for any $n \in \mathbb{N}$:

$$a^{b^n} = a^{f_{n-1}}(a^b)^{f_n},$$

where $(f_n)_{n\in\mathbb{N}}$ is the **Fibonacci sequence**,
and we use known results concerning the Fibonacci sequence,
for example the Cassini's identity:

$$f_{n-1}f_{n+1} - f_n^2 = (-1)^n.$$

In order to study the structure of S if

$$\langle S \rangle = \langle a, b \mid a^{b^2} = aa^b, aa^b = a^b a \rangle,$$

we notice that for any $n \in \mathbb{N}$:

$$a^{b^n} = a^{f_{n-1}}(a^b)^{f_n},$$

where $(f_n)_{n\in\mathbb{N}}$ is the **Fibonacci sequence**,
and we use known results concerning the Fibonacci sequence, for example the Cassini's identity:

$$f_{n-1}f_{n+1} - f_n^2 = (-1)^n.$$

In order to study the structure of S if

$$\langle S\rangle = \langle a, b \mid a^{b^2} = aa^b, aa^b = a^b a\rangle,$$

we notice that for any $n \in \mathbb{N}$:

$$a^{b^n} = a^{f_{n-1}}(a^b)^{f_n},$$

where $(f_n)_{n\in\mathbb{N}}$ is the **Fibonacci sequence**,
and we use known results concerning the Fibonacci sequence, for example the Cassini's identity:

$$f_{n-1}f_{n+1} - f_n^2 = (-1)^n.$$

$|S^2| = 3|S| - 2$ - A remark

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, A. Plagne, Y.V. Stanchescu, *Groups Geom. Dyn.*, 2017)

Let G be an ordered group and let S be a finite subset of G, $|S| \geq 4$. If

$$|S^2| = 3|S| - 2$$

then one of the following holds:

(1) $\langle S \rangle$ is an abelian group, at most 4-generated;

(2) $\langle S \rangle = \langle a, b \mid [a, b] = c, [c, a] = [c, b] = 1 \rangle$. In particular $\langle S \rangle$ is a nilpotent group of class 2;

(3) $\langle S \rangle = \langle a, b \mid a^b = a^2 \rangle$. Therefore $\langle S \rangle$ is the *Baumslag-Solitar group* $\mathcal{BS}(1, 2)$;

(4) $\langle S \rangle = \langle a \rangle \times \langle b, c \mid c^b = c^2 \rangle$;

(5) $\langle S \rangle = \langle a, b \mid a^{b^2} = aa^b, aa^b = a^b a \rangle$.

$|S^2| = 3|S| - 2$ - A remark

Corollary

Let G be an ordered group and let S be a finite subset of G, $|S| \geq 4$. If

$$|S^2| \leq 3|S| - 2,$$

then $\langle S \rangle$ is metabelian.

Corollary

Let G be an ordered group and let S be a finite subset of G, $|S| \geq 4$. If

$$|S^2| \leq 3|S| - 2$$

and $\langle S \rangle$ is nilpotent, then it is nilpotent of class at most 2.

$|S^2| = 3|S| - 2$ - A remark

Corollary

Let G be an ordered group and let S be a finite subset of G, $|S| \geq 4$. If

$$|S^2| \leq 3|S| - 2,$$

then $\langle S \rangle$ is metabelian.

Corollary

Let G be an ordered group and let S be a finite subset of G, $|S| \geq 4$. If

$$|S^2| \leq 3|S| - 2$$

and $\langle S \rangle$ is nilpotent, then it is nilpotent of class at most 2.

$|S^2| = 3|S| - 1$ - A result

Questions

Is there an orderable group with a finite subset S of order k (for any $k \geq 4$) such that $|S^2| = 3|S| - 1$

and $\langle S \rangle$ is non-metabelian (non-soluble)?

NO

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, A. Plagne, Y.V. Stanchescu, *Groups Geom. Dyn.*, 2017)

Let G be an ordered group, $\beta \geq -2$ any integer and let k be an integer such that $k \geq 2^{\beta+4}$. If S is a subset of G of finite size k and if

$$|S^2| \leq 3k + \beta,$$

then $\langle S \rangle$ is metabelian, and it is nilpotent of class at most 2 if G is nilpotent.

$|S^2| = 3|S| - 1$ - A result

Questions

Is there an orderable group with a finite subset S of order k (for any $k \geq 4$) such that $|S^2| = 3|S| - 1$

and $\langle S \rangle$ is non-metabelian (non-soluble)?

NO

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, A. Plagne, Y.V. Stanchescu, *Groups Geom. Dyn.*, 2017)

Let G be an ordered group, $\beta \geq -2$ any integer and let k be an integer such that $k \geq 2^{\beta+4}$. If S is a subset of G of finite size k and if

$$|S^2| \leq 3k + \beta,$$

then $\langle S \rangle$ is metabelian, and it is nilpotent of class at most 2 if G is nilpotent.

$|S^2| = 3|S| - 1$ - A result

Questions

Is there an orderable group with a finite subset S of order k (for any $k \geq 4$) such that $$|S^2| = 3|S| - 1$$ and $\langle S \rangle$ is non-metabelian (non-soluble)?

NO

Theorem (G.A. Freiman, M. Herzog, - , M. Maj, A. Plagne, Y.V. Stanchescu, *Groups Geom. Dyn.*, 2017)

Let G be an ordered group, $\beta \geq -2$ any integer and let k be an integer such that $k \geq 2^{\beta+4}$. If S is a subset of G of finite size k and if

$$|S^2| \leq 3k + \beta,$$

then $\langle S \rangle$ is metabelian, and it is nilpotent of class at most 2 if G is nilpotent.

An example

Example

For any $k \geq 3$, there exists an ordered group, with a subset S of finite size k, such that $\langle S \rangle$ is not soluble and

$$|S^2| = 4k - 5.$$

Let

$$G = \langle a \rangle \times \langle b, c \rangle,$$

where $\langle a \rangle$ is infinite cyclic and $\langle b, c \rangle$ is free of rank 2.
For any $k \geq 3$, define

$$S = \{a, ac, \cdots, ac^{k-2}, b\}.$$

Then

$$|S^2| = 4k - 5.$$

An example

Example

For any $k \geq 3$, there exists an ordered group, with a subset S of finite size k, such that $\langle S \rangle$ is not soluble and

$$|S^2| = 4k - 5.$$

Let

$$G = \langle a \rangle \times \langle b, c \rangle,$$

where $\langle a \rangle$ is infinite cyclic and $\langle b, c \rangle$ is free of rank 2.
For any $k \geq 3$, define

$$S = \{a, ac, \cdots, ac^{k-2}, b\}.$$

Then

$$|S^2| = 4k - 5.$$

Problems

Conjecture (G.A. Freiman)

If G is any torsion-free group, S a finite subset of G, $|S| \geq 4$, and

$$|S^2| \leq 3|S| - 4,$$

then S is contained in a geometric progression of length at most $2|S| - 3$.

Theorem (K.J. Böröczky, P.P. Palfy, O. Serra, *Bull. London Math. Soc.*, 2012)

The conjecture of Freiman holds if

$$|S^2| \leq 2|S| + \frac{1}{2}|S|^{\frac{1}{6}} - 3.$$

Problems

Conjecture (G.A. Freiman)

If G is any torsion-free group, S a finite subset of G, $|S| \geq 4$, and

$$|S^2| \leq 3|S| - 4,$$

then S is contained in a geometric progression of length at most $2|S| - 3$.

Theorem (K.J. Böröczky, P.P. Palfy, O. Serra, *Bull. London Math. Soc.*, 2012)

The conjecture of Freiman holds if

$$|S^2| \leq 2|S| + \frac{1}{2}|S|^{\frac{1}{6}} - 3.$$

Problems

Conjecture

If G is any torsion-free group, S a finite subset of G, $|S| \geq 4$, and

$$|S^2| \leq 3|S| - 2,$$

then $\langle S \rangle$ is metabelian and, if it is nilpotent, it is nilpotent of class 2.

P. Longobardi
Dipartimento di Matematica
Università di Salerno
via Giovanni Paolo II, 132, 84084 Fisciano (Salerno), Italy
E-mail address : plongobardi@unisa.it

Bibliography

- L. Babai, V.T. Sós, Sidon sets in groups and induced subgraphs of Cayley graphs, *Europ. J. Combinatorics*, **6** (1985), 101-114.
- Ja.G. Berkovich, G.A. Freiman, C.E. Praeger, Small squaring and cubing properties for finite groups, *Bull. Austral. Math. Soc.*, **44** (1991), 429-450.
- O. Bogopolski, Abstract commensurators of solvable Baumslag-Solitar groups, *Comm. Algebra*, **40(7) 44** (2012), 2494-2502.
- K.J. Böröczky, P.P. Palfy, O. Serra, O the cardinality of sumsets in torsion-free groups, *Bull. London Math. Soc.*, (2012), 1034-1041.
- R. Botto Mura, A. Rhemtulla, *Orderable Groups,* Lecture notes in pure and applied mathematics, **27** Marcel Dekker (1977).

Bibliography

- L.V. Brailovsky, G.A. Freiman, On a product of finite subsets in a torsion-free group, *J. Algebra* **130** (1990), 462-476.
- E. Breuillard and B. J. Green, Approximate groups, I: the torsion-free nilpotent case, *J. Inst. Math. Jussieu* **10** (2011), no. 1, 37-57.
- E. Breuillard, B.J. Green and T.C. Tao, The structure of approximate groups, *Publ. Math. IHES.* **116** (2012), 115-221.
- E. Breuillard, B.J. Green and T.C. Tao, A nilpotent Freiman dimension lemma, *European J. Combin.* **34** (8) (2013), 1287-1292.
- G.A. Freiman, On the addition of finite sets, *Izv. Vyss. Ucebn. Zaved. Matematika,* **6** (13) (1959), 202-213.
- G.A. Freiman, Inverse problems of additive number theory IV. On the addition of finite sets II, *(Russian) Elabuž. Gos. Ped. Inst. Učen. Zap.,* **8** (1960), 72-116.

Bibliography

- G.A. Freiman, Groups and the inverse problems of additive number theory, Number-theoretic studies in the Markov spectrum and in the structural theory of set addition (Russian), *Kalinin. Gos. Univ., Moscow* (1973), 175-183.
- G. A. Freiman, On two and three-element subsets of groups, *Aeq. Math.*, **22** (1981), 140-152.
- G.A. Freiman, *Foundations of a structural theory of set addition*, Translations of mathematical monographs, **v. 37**, American Mathematical Society, Providence, Rhode Island, 1973.
- G.A. Freiman, Structure Theory of Set Addition, *Astérisque*, **258** (1999), 1-33.
- G.A. Freiman, On finite subsets of nonabelian groups with small doubling, *Proc. of the Amer. Math. Soc.* **140**, no. 9 (2012), 2997-3002.

Bibliography

- G.A. Freiman, M. Herzog, P. Longobardi, M. Maj, Small doubling in ordered groups, *J. Austral. Math. Soc.*, **96** (2014), no. 3, 316-325.
- G.A. Freiman, M. Herzog, P. Longobardi, M. Maj, Y.V. Stanchescu, Direct and inverse problems in Additive Number Theory and in non-abelian group theory, *European J. Combin.*, **40** (2014), 42-54.
- G.A. Freiman, M. Herzog, P. Longobardi, M. Maj, Y.V. Stanchescu, A small doubling structure theorem in a Baumslag-Solitar group, *European J. Combin.*, **44** (2015), 106-124.
- G.A. Freiman, M. Herzog, P. Longobardi, M. Maj, Y.V. Stanchescu, Inverse problems in Additive Number Theory and in Non-Abelian Group Theory, *arXiv*:1303.3053 (2013), preprint, 1-31.

Bibliography

- G.A. Freiman, M. Herzog, P. Longobardi, M. Maj, Y.V. Stanchescu, Some inverse problems in group theory, *Note di Matematica*, **34** (2014), 89-104.
- G.A. Freiman, M. Herzog, P. Longobardi, M. Maj, A. Plagne, D.J.S. Robinson, Y.V. Stanchescu, On the structure of subsets of an orderable group, with some small doubling properties, *J. Algebra*, **445** (2016), 307-326.
- G.A. Freiman, M. Herzog, P. Longobardi, M. Maj, A. Plagne, Y.V. Stanchescu, Small doubling in ordered groups: generators and structures, *Groups Geom. Dyn.*, **11** (2017), 585-612.
- G.A. Freiman, M. Herzog, P. Longobardi, M. Maj, Y. V. Stanchescu, Small doubling in ordered nilpotent group of class 2, *European Journal of Combinatorics,* (2017) http://dx.doi.org/10.1016/j.ejc.2017.07.006., to appear.

Bibliography

G.A. Freiman, B.M. Schein, Interconnections between the structure theory of set addition and rewritability in groups, *Proc. Amer. Math. Soc.* **113** (1991), 899-910.

A. Geroldinger, I.Z. Ruzsa, *Combinatorial Number Theory and Additive Group Theory*, Birkäuser Verlag, Basel - Boston - Berlin, 2009.

A.M.W. Glass, *Partially ordered groups*, World Scientific Publishing Co., Series in Algebra, **v. 7**, 1999.

B. Green, What is ... an approximate group?, *Notices Amer. Math. Soc.* **59** (2012), no. 5, 655-656.

B. Green, I. Z. Ruzsa, Freiman's theorem in an arbitrary abelian group, *J. London Math. Soc.* **75** (2007), no. 1, 163-175.

Bibliography

- Y.O. Hamidoune, A.S. Lladó, O. Serra, On subsets with small product in torsion-free groups, *Combinatorica* (1998), no. 4, 529-540.
- Y.O. Hamidoune, A. Plagne, A generalization of Freiman's 3k–3 theorem, *Acta Arith.* **103** (2002), no. 2, 147-156.
- M. Herzog, P. Longobardi, M. Maj, On a combinatorial problem in group theory, *Israel J. Math* **82** (1993), 329-340.
- M.I. Kargapolov, Completely orderable groups, *Algebra i Logica (2)* **1** (1962), 16-21.
- J.H.B. Kemperman, On complexes in a semigroup, *Indag. Mat.* **18** (1956), 247-254.
- V.F. Lev, P.Y. Smeliansky, On addition of two distinct sets of integers, *Acta Arith.* **70** (1995), no. 1, 85-91.

Bibliography

- P. Longobardi, M. Maj, The classification of groups with the small squaring property on 3-sets, *Bull. Austral. Math. Soc.*, **46, 2** (1992), 263-269.
- A.I. Mal'cev, On ordered groups, *Izv. Akad. Nauk. SSSR Ser. Mat.* **13** (1948), 473-482.
- M.B. Nathanson, *Additive number theory - Inverse problems and geometry of sumsets*, Springer, New York, 1996.
- M.B. Nathanson, Inverse problems for linear forms over finite sets of integers, *J. Ramanujan Math. Soc.* **23** (2008), no. 2, 151-165.
- B.H. Neumann, On ordered groups, *Amer. J. Math.* **71** (1949), 1-18.
- A. Plagne, S. Tringali, Sums of dilates in ordered groups, *Comm. Algebra* **44** (2016), 5223-5236.

Bibliography

- Z. Ruzsa, Generalized arithmetic progressions and sumsets, *Acta Math. Hungar.* **65(4)** (1994), 379-388.
- I.Z. Ruzsa, An analog of Freiman's theorem in groups, *Astérisque* **258** (1999), 323-326.
- T. Sanders, The structure theory of set addition revisited, *Bull. Amer. Math. Soc.* **50(1)** (2013), 93-127.
- J.F. Sample, A. Shalev, Combinatorial Conditions in Residually Finite Groups, I, *J. Algebra* **157** (1993), no. 1, 43-50.
- A. Shalev, Combinatorial Conditions in Residually Finite Groups, II, *J. Algebra* **157** (1993), no. 1, 51-62.
- Y.V. Stanchescu, On addition of two distinct sets of integers, *Acta Arith.* **75** (1996), no. 2, 191-194.

Bibliography

- Y.V. Stanchescu, On the structure of sets with small doubling property on the plane, I, *Acta Arith.* **83** (1998), no. 2, 127-141.
- Y.V. Stanchescu, The structure of d-dimensional sets with small sumset, *J. Number Theory* **130** (2010), no. 2, 289-303.
- T.C. Tao, Product set estimates for non-commutative groups, *Combinatorica* **28(5)** (2008), 547-594.
- M.Tointon, Freiman's theorem in an arbitrary nilpotent group, *Proc. London Math. Soc.* **109** (2014), 318-352.

Thank you for the attention !