When is the lattice of closure operators on a subgroup lattice again a subgroup lattice?

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Groups St Andrews 2017

Definition

A partially ordered set L is a lattice if any two elements $x, y \in L$ have a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$.

Lattices

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Typical examples include the lattice of subsets of a set, or the lattice of subalgebras of an algebra (in the sense of universal algebra), ordered by inclusion.

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Question

Given a class of lattices, which ones are isomorphic to the lattice of subgroups of a group K?

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(iii) $\mathrm{cl}(\mathrm{cl}(x)) = \mathrm{cl}(x)$ *(idempotent).*

Comparing closure operators

If P is a partially ordered set, and ϕ, ψ are closure operators on P, we say $\phi \preceq \psi$ if and only if $\phi(x) \leq \psi(x)$ for all $x \in P$.

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If P is a lattice, then the set of closure operators on P is again a lattice, via

$$
\begin{aligned}
(\phi \wedge \psi)(x) &= \phi(x) \wedge \psi(x)\\
(\phi \vee \psi)(x) &= \bigwedge \{\nu(x) \mid \phi \leq \nu \text{ and } \psi \leq \nu\}
\end{aligned}
$$

Question

Let G be a group: $\mathrm{sub}(G)$ is the lattice of all subgroups of G.

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Question

Let G be a group. When is $\mathrm{c.o}(\mathrm{sub}(G))$ isomorphic to $\mathrm{sub}(K)$ for some group K?

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The lattice of subgroups of a group is always algebraic. This reflects the fact that a group is completely determined by its finitely generated subgroups.

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Definition

Let G be a group. A closure operator ϕ on $\mathrm{sub}(G)$ is algebraic if for every $H \leq G$, we have

$$\phi(H) = \bigvee_{\substack{K \leq H \\ K \text{ f.g.}}} \phi(K).$$

Stick to algebraic closure operators

Theorem (Kilpack)

The lattice of all algebraic closure operators on a lattice L is an algebraic lattice; it is a lower subsemilattice of the lattice of all closure operators on L.

The new question

Question

Let G be a group. When is the lattice of all algebraic closure operators on $\mathrm{sub}(G)$, $\mathrm{aco}(\mathrm{sub}(G))$, isomorphic to $\mathrm{sub}(K)$ for some group K?

Easy case for yes

Theorem

Let G be a cyclic group C_{p^n} for some prime p. Then $\mathrm{aco}(\mathrm{sub}(G))$ is isomorphic to $\mathrm{sub}(C_{q_1\cdots q_n})$, where $q_1 < \cdots < q_n$ are distinct primes.

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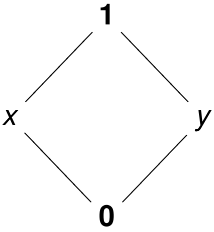
This corresponds to the subgroup of subsets of $\{1, p, p^2, \ldots, p^{n-1}\}$. The subgroups of $C_{q_1\cdots q_n}$ correspond to divisors of $q_1 \cdots q_n$, which correspond to subsets of $\{q_1, \ldots, q_n\}$. Hence $\mathrm{aco}(\mathrm{sub}(C_{p^n})) \cong \mathrm{sub}(C_{q_1\cdots q_n})$. $\square$

Closure operators on $\mathrm{sub}(C_{pq})$

Lattice



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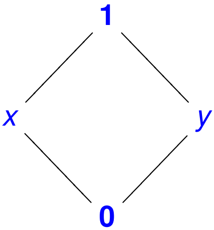
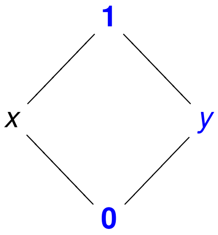


Closure operators

$\phi_{\mathbf{0},x,y,\mathbf{1}}$

Lattice



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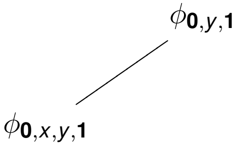

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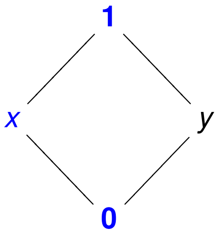

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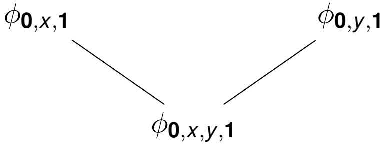

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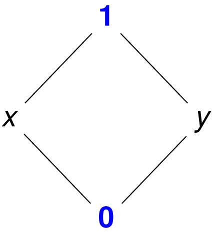

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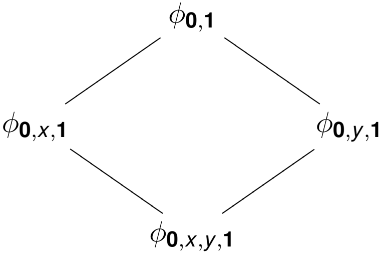

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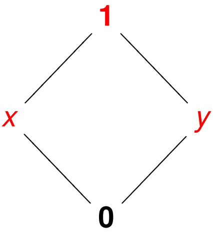

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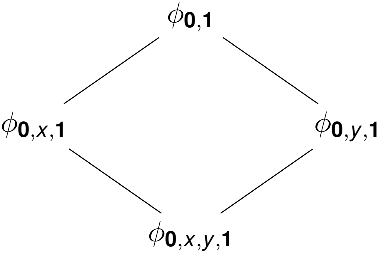

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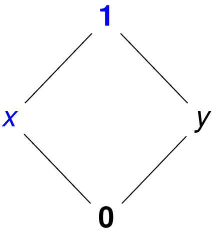

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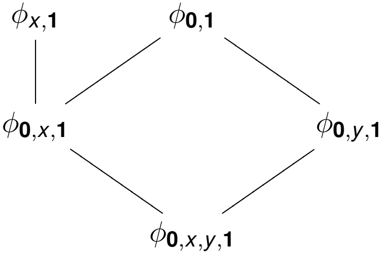

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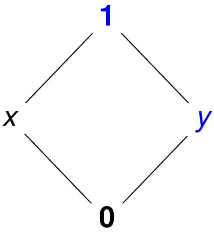

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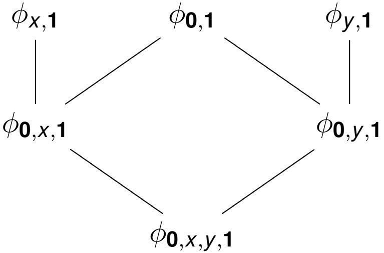

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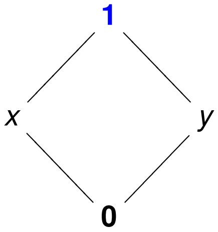

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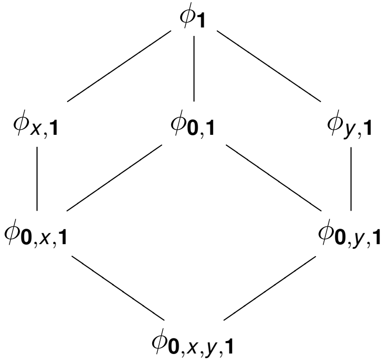

Proposition

If G has a unique maximal subgroup, then G is cyclic of prime power order.

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Two easy exercises

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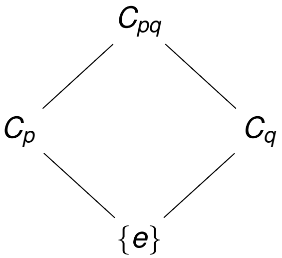
Proposition

If G has exactly two maximal subgroups, then G is cyclic of order $p^a q^b$ for some distinct primes p and q, $a, b > 0$.

Is this a subgroup lattice?

Subgroups



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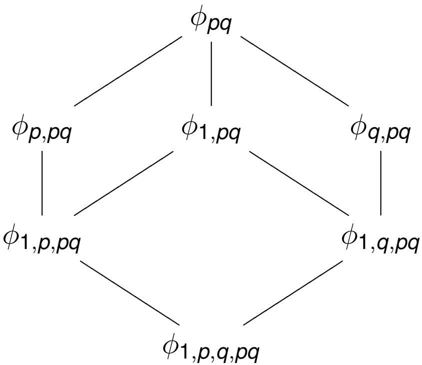

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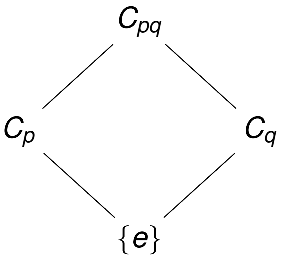

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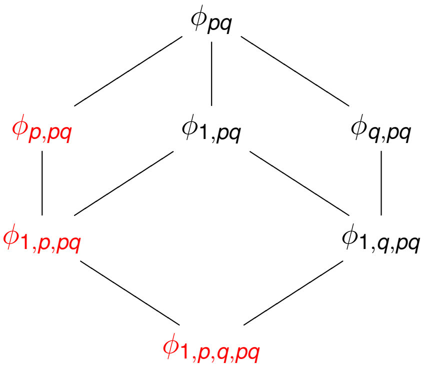

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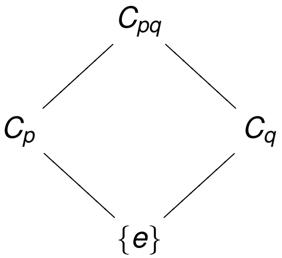

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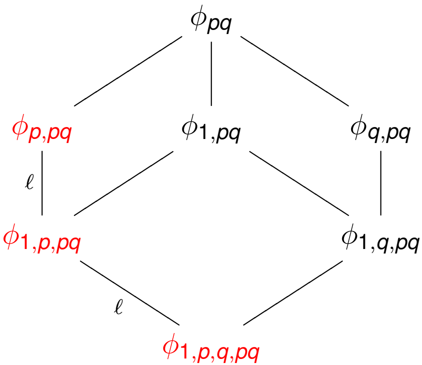

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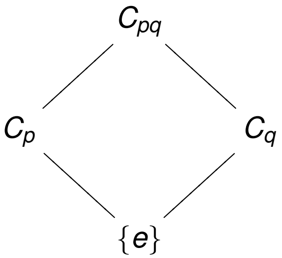

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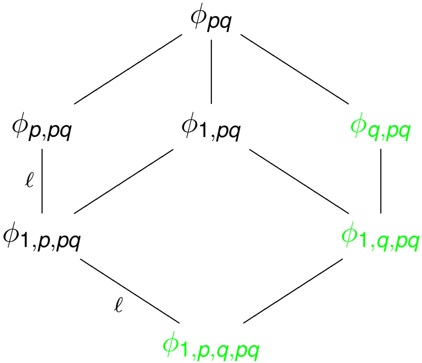

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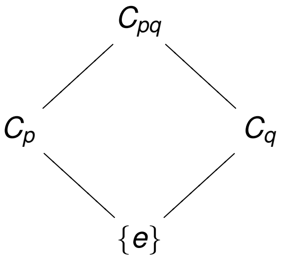

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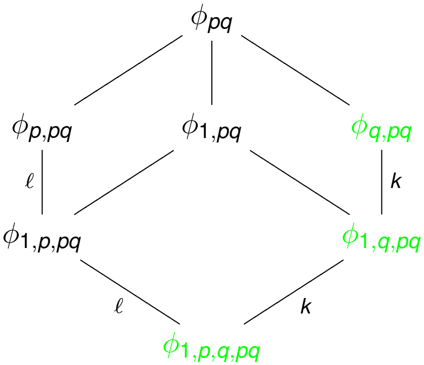

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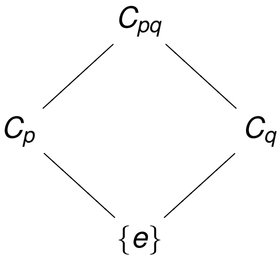

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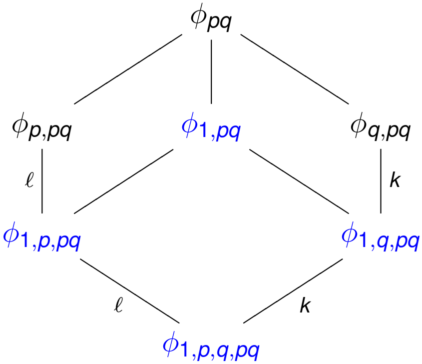

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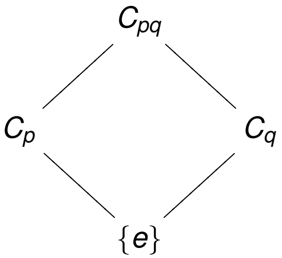

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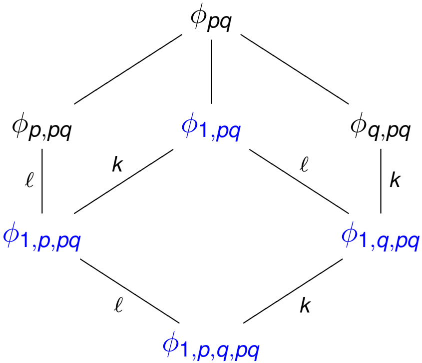

Is this a subgroup lattice?

Subgroups

$$
\begin{tikzcd}
 & C_{pq} \arrow[dl, dash] \arrow[dr, dash] & \\
C_p \arrow[dr, dash] & & C_q \arrow[dl, dash] \\
 & \{e\} &
\end{tikzcd}
$$

Closure operators

$$
\begin{tikzcd}
 & & \phi_{pq} \arrow[dll, dash, "\text{prime}"'] \arrow[d, dash, "\text{prime}"] \arrow[drr, dash, "\text{prime}"] & & \\
\phi_{p,pq} \arrow[d, dash, "\ell"'] & & \phi_{1,pq} \arrow[dll, dash, "k"'] \arrow[drr, dash, "\ell"] & & \phi_{q,pq} \arrow[d, dash, "k"] \\
\phi_{1,p,pq} \arrow[drr, dash, "\ell"'] & & & & \phi_{1,q,pq} \arrow[dll, dash, "k"] \\
 & & \phi_{1,p,q,pq} & &
\end{tikzcd}
$$

Not a subgroup lattice

Theorem

Let p and q be distinct primes. Then the lattice of (algebraic) closure operators on $\mathrm{sub}(C_{pq})$ is not isomorphic to a subgroup lattice.

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The key constructions

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Given a subgroup H of prime order, let $\phi(K) = \langle K, H \rangle$.

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Given a subgroup H of prime order, let $\phi(K) = \langle K, H \rangle$. Then let ψ map $\{e\}$ to itself, and K to $\langle K, H \rangle$ otherwise.

The key constructions

Idea: Find closure operators ϕ and ψ with $\psi < \phi$, and such that for all η, $\eta < \phi \implies \eta \leq \psi$. If the closure operator lattice is a subgroup lattice, then $\downarrow(\phi)$ must be a finite chain.

Given a subgroup H of prime order, let $\phi(K) = \langle K, H\rangle$. Then let ψ map $\{e\}$ to itself, and K to $\langle K, H\rangle$ otherwise. These two satisfy the condition above.

Main exclusion theorem

Theorem (Main exclusion theorem)

Suppose that G is a group, H is a subgroup of prime order, and there exist subgroups M and N such that:

- *$H \not\leq M$ and $H \not\leq N$; and*
- *M and N are incomparable;*

then $\mathrm{aco}(\mathrm{sub}(G))$ is not a subgroup lattice.

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then $\mathrm{aco}(\mathrm{sub}(G))$ is not a subgroup lattice.

Corollary

If G has at least three distinct subgroups of prime order, then $\mathrm{aco}(\mathrm{sub}(G))$ is not a subgroup lattice.

Quotients

Theorem

Let $N \lhd G$ be a finitely generated normal subgroup. Define

$$\phi_N(K) = \begin{cases} G & \textit{if } N \leq K; \\ K & \textit{otherwise.} \end{cases}$$

Then $\downarrow(\phi_N)$ *in* $\mathrm{aco}(\mathrm{sub}(G)) \cong \mathrm{aco}(\uparrow(N)$ *in* $\mathrm{sub}(G))$*.*
In particular, if $\mathrm{aco}(\mathrm{sub}(G))$ *is a subgroup lattice, then* $\mathrm{aco}(\mathrm{sub}(G/N))$ *is a subgroup lattice.*

Finite case

Theorem

Let G be a finite group. Then $\operatorname{aco}(\operatorname{sub}(G))$ is a subgroup lattice if and only if G is a cyclic group of prime power order.

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Sketch. Must have order p^n or $p^a q^b$, and either exactly one or exactly two subgroups of prime order. Use induction on n and on $a+b$.

Infinite mixed case

Theorem

Let G be a group that has nontrivial elements of finite and of infinite order. Then $\mathrm{aco}(\mathrm{sub}(G))$ is not a subgroup lattice.

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Proof. Let H be a subgroup of prime order, and let x be an element of infinite order.

Infinite mixed case

Theorem

Let G be a group that has nontrivial elements of finite and of infinite order. Then $\mathrm{aco}(\mathrm{sub}(G))$ *is not a subgroup lattice.*

Proof. Let H be a subgroup of prime order, and let x be an element of infinite order.
Then $M = \langle x^2 \rangle$ and $N = \langle x^3 \rangle$ are incomparable, and do not contain H. By the Main Exclusion Theorem, $\mathrm{aco}(\mathrm{sub}(G))$ is not a subgroup lattice. $\square$

Infinite torsion group

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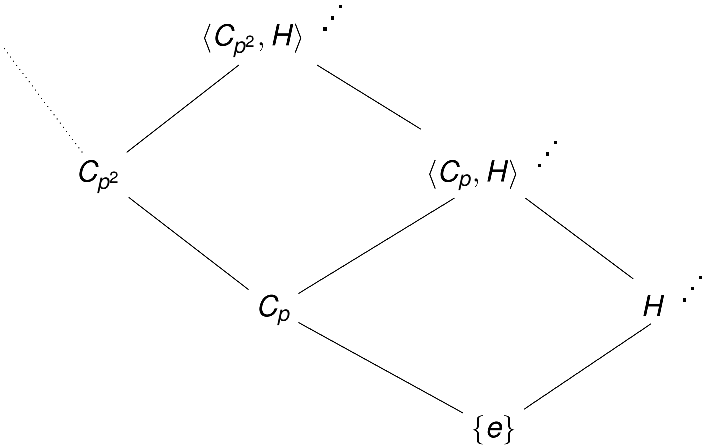
If G is a p-group, one can show it has a unique subgroup of order p^n for all n, so $G \cong \mathbf{Z}_{p^\infty}$.

Otherwise, we can reduce to the case where G has a subgroup isomorphic to $\mathbf{Z}_{p^\infty}$ and a subgroup H of order $q \neq p$.

An infinite chain



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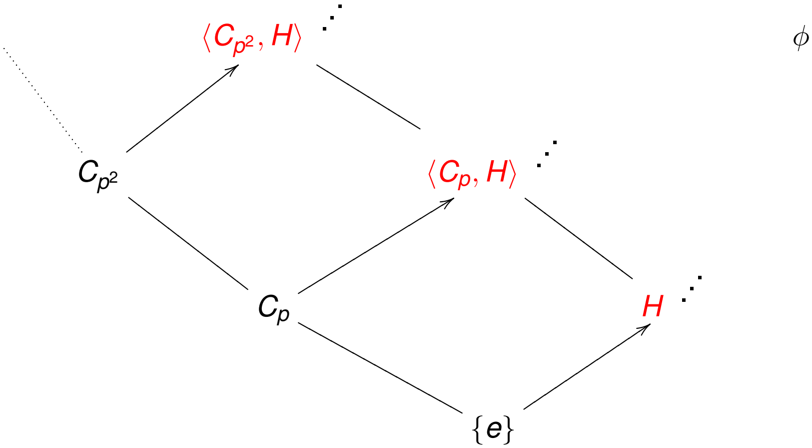

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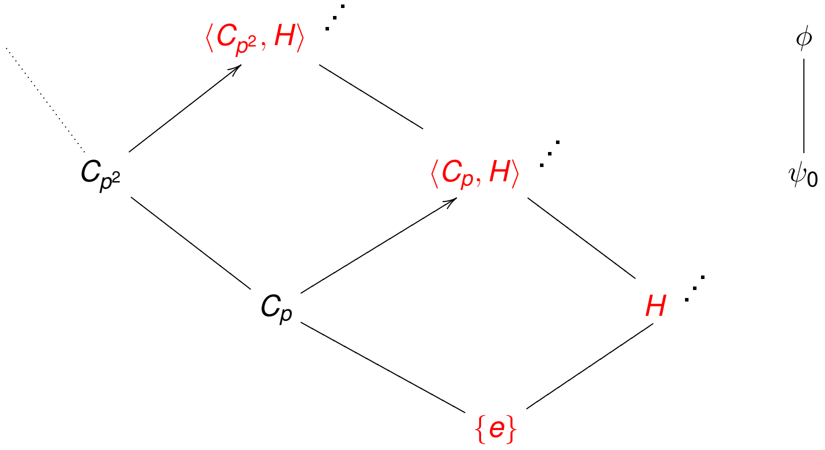
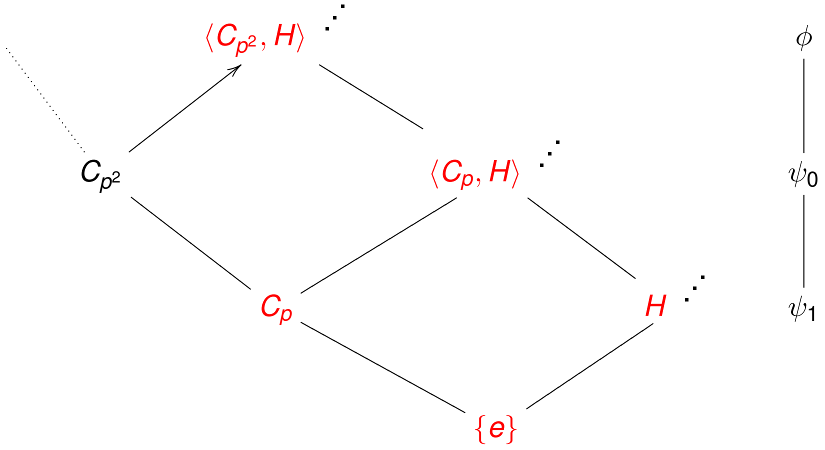
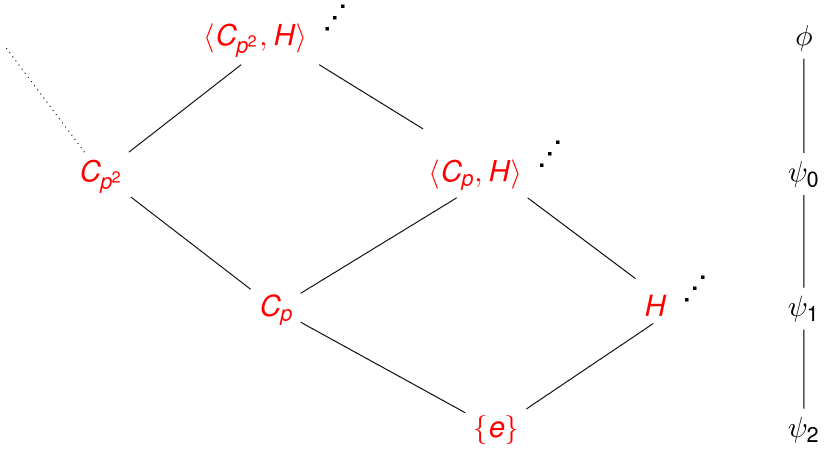


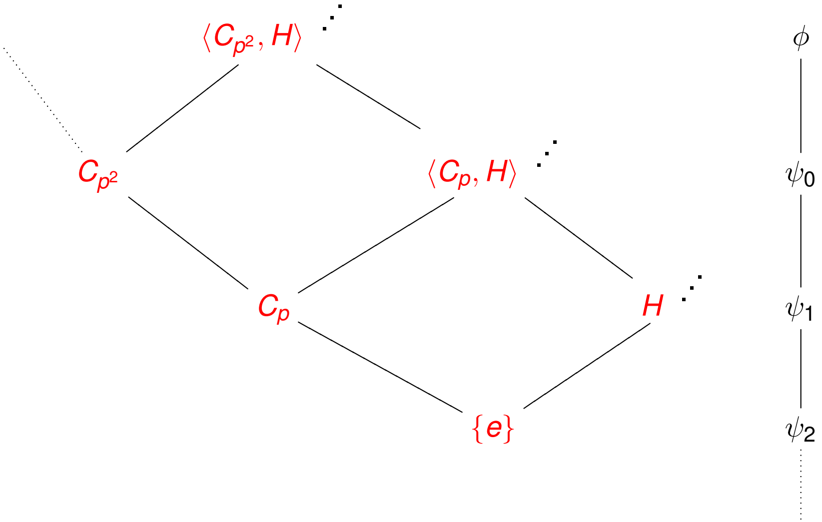
An infinite chain



An infinite chain



An infinite chain



Groups with torsion

Theorem

Let G be a group (finite or infinite) that has a nontrivial element of finite order. Then $\mathrm{aco}(\mathrm{sub}(G))$ is a subgroup lattice if and only if G is cyclic of prime power order or isomorphic to the Prüfer p-group $\mathbb{Z}_{p^\infty}$.

Open cases

What about torsionfree groups?

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Most of the constructions require a nontrivial minimal subgroup.

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If G has a nontrivial finitely generated normal abelian subgroup, then $\mathrm{aco}(\mathrm{sub}(G))$ is not a subgroup lattice.

If G has a finitely generated normal subgroup N, then G/N must be torsionfree, cyclic of prime power order, or isomorphic to $\mathbb{Z}_{p^\infty}$.

Conjecture

Conjecture

Let G be a group. Then $\mathrm{aco}(\mathrm{sub}(G))$ is a subgroup lattice if and only if G is cyclic of prime power order of isomorphic to the Prüfer p-group.

Thank you for your attention!