Zassenhaus Conjecture for small groups

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Groups St Andrews 2017
Birmingham
August 6th - 12th 2017

Notation and Setting

- G finite group,
- $\mathbb{Z}G$ integral group ring over G,
- Augmentation map: $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$, $\varepsilon(\sum\limits_{g \in G} z_g g) = \sum\limits_{g \in G} z_g$,
- $\mathrm{V}(\mathbb{Z}G)$ group of units of augmentation 1 aka normalized units.
- All units of $\mathbb{Z}G$ are of the form $\pm\mathrm{V}(\mathbb{Z}G)$, so it suffices to consider $\mathrm{V}(\mathbb{Z}G)$.
- Exponents of G and $\mathrm{V}(\mathbb{Z}G)$ coincide. (Cohn-Livingstone '67)

Basic question:

How is the torsion part of $\mathrm{V}(\mathbb{Z}G)$ connected to G?

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Main open question concerning torsion units:

(First) Zassenhaus Conjecture (H.J. Zassenhaus, 1974)

(ZC) Any $u \in V(\mathbb{Z}G)$ of finite order is conjugate in $\mathbb{Q}G$ to an element $g \in G$. I.e. there exists a unit $x \in \mathbb{Q}G$ such that $x^{-1}ux = g$.

If such x and g exist, one says that u is **rationally conjugate** to g.

The Zassenhaus Conjecture is known to hold for

- Nilpotent Groups (Weiss '91)
- Groups with normal Sylow subgroup with abelian complement (Hertweck '07)
- Cyclic-By-Abelian Groups (Caicedo-M'-del Rio '13)
- Some other special series of nilpotent-by-abelian groups
- Groups till order 192 (this talk)
- Some special (mostly small) non-solvable groups (Luthar-Passi, Hertweck, Gildea, Kimmerle-Konovalov, Bächle-M', M'-Serrano-del Río)

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Partial Augmentations

Let x^G be a conjugacy class in G and $u = \sum_{g \in G} z_g g \in \mathbb{Z}G$. Then

$$\varepsilon_x(u) = \sum_{g \in x^G} z_g$$

is called the **partial augmentation** of u with respect to x.

Theorem (Marciniak-Ritter-Sehgal-Weiss '87)

$u \in V(\mathbb{Z}G)$ is rationally conjugate to an element of G if and only if $\varepsilon_x(u^d) \geq 0$ for all $x \in G$ and $d \in \mathbb{Z}$.

Theorem (Higman '39, Berman '53)

If $u \in V(\mathbb{Z}G)$ is a torsion unit, then $\varepsilon_1(u) = 0$ or $u = 1$.

Theorem (Hertweck '07)

Let u be a torsion unit in $V(\mathbb{Z}G)$ and $x \in G$ s.t. $\varepsilon_x(u) \neq 0$. Then the order of x divides the order of u.

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Methods: HeLP (Hertweck-Luthar-Passi)

A standard character theoretic method to study torsion units in $\mathrm{V}(\mathbb{Z}G)$ is known as HeLP (**He**rtweck**L**uthar**P**assi).

$u \in \mathrm{V}(\mathbb{Z}G)$ torsion, χ (ordinary) character of G. By linear extension χ is a character of $\mathrm{V}(\mathbb{Z}G)$.
ψ a character of $\langle u \rangle$. Then

$$\langle \chi|_{\langle u \rangle}, \psi \rangle \in \mathbb{Z}_{\geq 0}.$$

This is an expression linear in the partial augmentations of $\langle u \rangle$, since $\chi(u^i) = \sum_{x^G} \varepsilon_x(u^i)\chi(x)$, the sum running over conjugacy classes in G.
$\rightsquigarrow$ System of linear inequalities with finitely many integral solutions.

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$\rightsquigarrow$ System of linear inequalities with finitely many integral solutions.

The solutions correspond to the partial augmentations of units in $\mathbb{C}G$ of order $\circ(u)$ having integral partial augmentations.
If all solutions are trivial, i.e. one partial augmentation is 1 and the others 0, then units of order $\circ(u)$ in $\mathrm{V}(\mathbb{Z}G)$ are rationally conjugate to elements in G.

In any case we are left with **finitely many** possible partial augmentations for a possible counterexample. For each of them we can compute the eigenvalues under any representation of G.

This procedure is available in a `GAP`-package (Bächle-M').

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Methods: Quotients

$N \trianglelefteq G$, $u \in \mathrm{V}(\mathbb{Z}G)$ with specific partial augmentations.

$\varphi : \mathbb{Z}G \to \mathbb{Z}(G/N)$ natural homomorphism. Then the partial augmentations of $\varphi(u)$ are given by sums of partial augmentations of u which correspond to conjugacy classes of G fusing in G/N.

$\rightsquigarrow$ If we know that no unit with augmentations of $\varphi(u)$ exists in $\mathrm{V}(\mathbb{Z}(G/N))$, then u does not exist.

Methods: Partially central units (Höfert, Herman-Singh)

D irreducible $\mathbb{C}$-representation of $G \leftrightarrow e \in \mathbb{C}G$ idempotent.
$u \in \mathrm{V}(\mathbb{Z}G)$ with specific partial augmentations given, then eigenvalues of $D(u)$ can be computed.

If $ue \in \mathbb{C}Ge$ is central, conjugation will not change the ue-part of $u = ue + u(1-e)$.
Show $ue \notin \mathbb{Z}Ge$ using a $\mathbb{Z}$-basis from $\{D(g) | g \in G\}$.
$\rightsquigarrow u \notin \mathbb{Z}G$.

Implemented as `GAP`-routine (Herman-Konovalov-Singh).

Methods: Additional arguments from literature

More restrictions can be obtained from some specific results in the literature. Of particular importance are the following results of Hertweck.

Theorem (Hertweck)

Let $N \trianglelefteq G$ be a p-group and $u \in \mathrm{V}(\mathbb{Z}G)$ a torsion unit mapping to 1 under $\mathbb{Z}G \to \mathbb{Z}(G/N)$. Then u is conjugate in the p-adic group ring $\mathbb{Z}_p G$ to an element of G.

Lemma (Hertweck '07)

Let $u \in \mathrm{V}(\mathbb{Z}G)$ be a torsion unit such that the p-part of u is conjugate in $\mathbb{Z}_p G$ to an element x in G. Then $\varepsilon_g(u) \neq 0$ implies that the p-part of g is conjugate to x in G.

Double action lattices (work ongoing)

Existence of u is equivalent to existence of $\mathbb{Z}(G \times \langle u \rangle)$-lattice M which is

- $\mathbb{Z}G$-free of rank 1 (like $\mathbb{Z}G$),
- the action of u on the G fix-points in M is trivial (u is normalized)
- $\mathbb{C} \otimes_{\mathbb{Z}} M \cong M(u)$. Where $M(u)$ is a $\mathbb{C}(G \times \langle u \rangle)$-module, existing by the HeLP-method, such that

$$(g, u^i)m = gmu^{-i}, \;\; m \in M(u), g \in G, i \in \mathbb{Z}.$$

Determining the existence of such a lattice is a finite problem, but not doable in practice.
Changing to the semilocal coefficient ring $\mathbb{Z}_{(\circ(u))}$ this becomes computable by GAP-routines implemented by F. Eisele.
$\rightsquigarrow$ Computations are complicated and not always straightforward, but promising.

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Thank you for your attention !