Code algebras, axial algebras and VOAs

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Joint work with Alonso Castillo-Ramirez (University of Guadalajara) and Felix Rehren

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 - Mathematicians noticed some intriguing links between finite groups and modular functions, two apparently unrelated mathematical objects dubbed Monstrous Moonshine. This led to the moonshine VOA V[‡].
 - Code VOAs are an important class where a binary linear code governs the representation theory of the VOA.
 - All framed VOAs V (such as V^{\$}) have a unique code sub VOA and V is a simple current extension of its code sub VOA.

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 - Algebras generated by idempotents *a* whose adjoint acts semi-simply on the algebra. This gives a decomposition

$$A = A_1 \oplus A_0 \oplus A_{\lambda_1} \oplus \cdots \oplus A_{\lambda_k}$$

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where A_{λ} is the λ -eigenspace for ad_a .

 All the idempotents in the given generating set satisfy the same set of fusion rules which are a table of where the product of an element of A_λ with an element of A_μ lies.

• We get interesting non-associative algebras!

Definition

Let $C \subset \mathbb{F}_2^n$ be a binary linear code of length n, \mathbb{F} be a field of characteristic 0 and $a, b, c \in \mathbb{F}$.

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where $C^*:=C-\{0,1\}$, modulo the relations

$$t_{i} \cdot t_{j} = \delta_{i,j}$$

$$t_{i} \cdot e^{\alpha} = \begin{cases} a e^{\alpha} & \text{if } \alpha_{i} = 1 \\ 0 & \text{if } \alpha_{i} = 0 \end{cases}$$

$$e^{\alpha} \cdot e^{\beta} = \begin{cases} b e^{\alpha + \beta} & \text{if } \alpha \neq \beta, \beta^{c} \\ c \sum_{i \in \text{supp}(\alpha)} t_{i} & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha = \beta^{c} \end{cases}$$

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Theorem

Let C be a binary linear code such that one can build a code VOA V_C . Then, the code algebra $A_C(\frac{1}{4}, b, 4b^2)$ embeds in V_C .

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Let A_C be a non-degenerate code algebra.

- If C = {0, 1, α, 1 + α}, then A_C has exactly two non-trivial proper ideals,
- otherwise A_C is simple.

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Code algebras have a large automorphism group which contains a group of the form M:Aut(C), where M is generated by involutions coming from some idempotents.

Frobenius form

Definition

A Frobenius form on a code algebra is a symmetric bilinear form $(\cdot,\cdot):A\times A\to \mathbb{F}$ such that

• the form associates. That is, (x, yz) = (xy, z) for all $x, y, z \in A$.

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Theorem

Let A be a non-degenerate code algebra. Then A admits a unique Frobenius form (up to scaling) and it is given by:

$$(t_i, t_j) = \delta_{i,j}$$

 $(t_i, e^{\alpha}) = 0$
 $(e^{\alpha}, e^{\beta}) = \frac{c}{a} \delta_{\alpha,\beta}$

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A code algebra A has some obvious idempotents, namely the t_i .

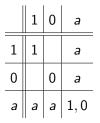
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Lemma (s-map construction)

There exists an idempotent

$$\mathfrak{s}(D,\mathbf{v}):=\lambda\sum_{i\in\mathrm{supp}(D)}t_i+\mu\sum_{lpha\in D^*}(-1)^{(\mathbf{v},lpha)}e^{lpha}$$

where $\lambda, \mu \in \mathbb{F}$ satisfy a linear and quadratic equation, respectively.

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$$e_{\pm} := \lambda \sum_{i \in \operatorname{supp}(lpha)} t_i \pm \mu e^{lpha}$$

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Corollary

A non-degenerate code algebra is generated by idempotents if $ac > \frac{c}{2|\alpha|}$.

Theorem

If ac $> \frac{c}{2|\alpha|}$ and $a \neq \frac{1}{3|\alpha|}$ then the small idempotents exist, are primitive and semi-simple, and have fusion rules given by

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	1	0	λ	$\frac{2\lambda-1}{2}$	$ u_+ $	ν_
1	1		λ	$\frac{2\lambda-1}{2}$	$ u_+ $	ν_
0		0			$ u_+ $	ν_{-}
λ	λ		$1,\lambda,rac{2\lambda-1}{2}$		$ u_{-}$	ν_+
$\frac{2\lambda-1}{2}$	$\frac{2\lambda-1}{2}$			$1, \frac{2\lambda - 1}{2}$	$ u_+ $	ν_
ν_+	ν_+	ν_+	$ u_{-}$	ν_+	$1,0,\lambda,rac{2\lambda-1}{2}, u_+$	$0, \lambda$
ν_{-}	ν_	ν_{-}	ν_+	ν_{-}	$0,\lambda$	$1,0,\lambda,rac{2\lambda-1}{2}, u$
where $ u_{\pm} := rac{1}{4} \pm \mu b.$						

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Axial algebras

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Corollary

Let C be a simplex or first order Reed-Muller code and $A_C(a, b, c)$ be a non-degenerate code algebra with $ac > \frac{c}{2|\alpha|}$ and $a \neq \frac{1}{3|\alpha|}$. Then, A is an axial algebra.

Let $C = H_8$ be the extended Hamming code and $(a, b, c) = (\frac{1}{4}, \frac{1}{2}, 1)$.

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The *s*-map construction gives two additional sets of eight mutually orthogonal idempotents, s(C, v), one for v with odd weight, one for v even weight.

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Moreover, these have the same fusion rules as the t_i making A_{H_8} an axial algebra (of Jordan type). And it has automorphism group containing

 2^{6} : (*PSL*₃(2) × *S*₃).

Thank you for listening!

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