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On Semigroups Admitting Conjugates

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Introduction

While every subsemigroup of a group is cancellative, a famous theorem of A.I. Mal'cev (1939) shows that not every cancellative semigroup is embeddable in a group. Patterned after the classical quotient construction, Oyestein Ore (1931) discovered the ”principle of common left multiple” to embed a non-commutative domain into a division ring. Using this as a backdrop, Malcev, B.H. Neumann and Taylor developed semigroup equivalents of nilpotent groups of class n and proved that cancellative semigroups of nilpotent class n are embeddable in groups of the same nilpotency class. In this talk, we investigate some equational classes of semigroups admitting conjugates - and prove that all the valid group theory implications do carry over to the equational theory of semigroups admitting conjugates.

Basic Definitions, Facts and Notations

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- If for each $x, y \in S$ there exist an element $z \in S$ such that $xy = yxz$, then z is called commutator of x and y, and we say S admits commutators.

Basic Definitions, Facts and Notations

Fact: If S is a cancellative semigroup such that for $x, y \in S$, both conjugate of x by y and commutator of x and y exist, then both conjugate and commutator are unique.

Notations:

- Conjugate of x by y is denoted by x^y.
- Commutator of x and y is denoted by $[x, y]$.
- By $[x, y, z]$ we mean $[[x, y]\,,\ z]$.

Basic Definitions, Facts and Notations

- ☞ Let S be a cancellative semigroup which admits conjugates. If for all elements x, y and z in S,

$$xyzyx = yxzxy\,,$$

then S is called nilpotent of class 2.

- ☞ Fact: Let S be a cancellative semigroup which admits commutators. Then S is nilpotent of class 2 if and only if

$$z[x,y] = [x,y]z\,, \text{for all elements } x,\ y \text{ and } z \text{ in } S.$$

☞ Fact: If a cancellative semigroup S admits commutators then it must admit conjugates as well.

In fact since $xy = yx[x,y]$ so x^y exist and $x^y = x[x,y]$.

Moreover
$$xy = y\,x^y \qquad (*)\,.$$

Examples

In $GL_2(\mathbf{R})$, let $S_1 = \left\{ \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \;\middle|\; a, b \in \mathbf{I},\ b \neq 0 \right\}$ and $S_2 = \left\{ \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \;\middle|\; a, b \in \mathbf{R},\ 0 < b < 1 \right\}$. Then both S_1 and S_2 are cancellative semigroups and admit conjugates. In fact for any $X = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & c \\ 0 & d \end{pmatrix}$ in S_1 or S_2, $X^Y = \begin{pmatrix} 1 & c + ad - bc \\ 0 & b \end{pmatrix}$ is in both S_1 and S_2.

But $[X, Y] = \begin{pmatrix} 1 & c + ad - bc - a \\ 0 & 1 \end{pmatrix}$ is in S_1 but not in S_2.

Therefore S_1 is a cancellative semigroup that admits both conjugates and commutators and S_2 is a cancellative semigroup that admits conjugates but not commutators.

Embedding of Semigroups admitting Conjugates

Background : In general cancellative semigroups are not embeddable in groups due to A.I. Mal'cev (1939).

Definition Let S be a cancellative semigroup which admits conjugates. For any elements a, b, c and d in S we define

- $a \setminus b = \{(x, y)| \ ay = xb^y, \ \ x, y \in S\}$.

- The set of all $a \setminus b$ is denoted by $\overline{S}$, i.e.

$$\overline{S} = \{a \setminus b \mid \ a, b \in S\},$$

- In $\overline{S}$ we define binary operation $*$ as

$$(a \setminus b) * (c \setminus d) = ac \setminus db^c$$

Lemma : Let S be a cancellative semigroup which admits conjugates. Then for a, b, c, x, y, z, u, and v in S:

1. $x^x = x$,
2. $(x^y)^z = x^{yz}$,
3. $(xy)^z = x^z y^z$,
4. If $ay = xb^y, cy = xd^y, av = ub^v$, then $cv = ud^v$ (An analog of Ore's condition),
5. If $(a \searrow b) \cap (c \searrow d) \neq \varnothing$, then $a \searrow b = c \searrow d$,
6. $a \searrow a = b \searrow b$,
7. $au \searrow bu = a \searrow b$,
8. $ua \searrow bu^a = a \searrow b$,
9. $au \searrow u = av \searrow v$.

Theorem 1: Let S be a cancellative semigroup which admits conjugates. Then $(\overline{S}, *)$ is a group and S is embeddable into $\overline{S}$.

In 1942, F. Levi proved that a group satisfies the commutator law $[[x, y], z] = [x, [y, z]]$ if and only if the group is of nilpotent of class at most 2. By a classical result of Mal'cev (also, independently by Neumann and Taylor), a cancellation semigroup satisfies the semigroup law $xyzyx = yxzxy$ if and only it is a subsemigroup of a **group** of nilpotent class at most 2. Here we prove an analog of Levi's theorem for conjugates by characterizing semigroups embeddable in groups of nilpotent of class 2 by means of a single conjugacy law.

Theorem 2: Let S be a cancellative semigroup which admits conjugates, then S is nilpotent of class 2 if and only if it satisfies the conjugacy law $x^{y^z} = x^y$.

Corollary: Let S be a cancellative semigroup which admits conjugates, then S is nilpotent of class 2 if and only if it satisfies the conjugacy law $x^{yz} = x^{zy}$.

Following Mal’cev, B.H. Neumann and Taylor, we define a semigroup S to be nilpotent of class 3 if it satisfies the law

$$(xyzyx)u(yxzxy) = (yxzxy)u(xyzyx)\,;$$

and inductively we say S is of nilpotent class n if it satisfies the law $fug = guf$ where the law $f = g$ defines semigroups of nilpotent class $n-1$ and u is a new variable not occurring in the terms f or g.

Theorem 3: Let S be a cancellative semigroup which admits conjugates, then S is nilpotent of class n if and only if it satisfies the $(n+1)$-variable conjugacy law $x^f = x^g$ where x is a variable not occurring in the terms f or g.

Proof: Assume that S satisfies the law $x^f = x^g$. Let $x = fu$ where u is a new variable, then since $x^f = (fu)^f = fu^f = uf$ so must $uf = (fu)^g$. Therefore $fug = (fu)g = g(fu)^g = g(uf) = guf$ which means S is nilpotent of class n.
Conversely assume that S is nilpotent of class n that is $fug = guf$ is a law. Then by the very definition of conjugates, we have $xf = fx^f$. Premultiplying both sides of this equation by gy, where y is a new variable, we get $gyxf = gyfx^f$. Using the nilpotent identity $fug = guf$ twice, we obtain $fyxg = fygx^f$. Left canceling the common term fy we get $xg = gx^f$. But $xg = gx^g$, therefore $gx^g = gx^f$. Finally left canceling the common term g, we obtain the desired conjugacy law $x^f = x^g$.

Semigroups admitting Commutators

Theorem Let S be a cancellative semigroup which admits commutators. The following conditions are equivalent for all x, y and z in S:

(a) $[x,y]z = z[x,y]$ -nilpotent of class 2;

(b) $[x,y,z] = [x,[y,z]]$ -associativity of the commutators ;

(c) $[xy,z] = [x,z][y,z]$ -distributivity of the commutators ;

(d) $xyzyx = yxzxy$;

(e) $x^{y^z} = x^y$.

Thank you !