On p-groups of conjugate rank 1 and nilpotency class 3.

Tushar Kanta Naik
HRI, Allahbad, India
(Joint work with Rahul Kitture and Manoj Yadav)

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- **Theorem(J.Cossey-T.O.Hawkes,2000)**
  Let p be a prime and $0 = e_o < e_1 < \cdots < e_n$ be integers. Then there exists a p-group G with nilpotency class 2 such that, the set of conjugacy class sizes of G is exactly $\{1 = p^{e_0}, p^{e_1}, \ldots, p^{e_n}\}$.

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- **Problem 1(Avinoam Mann,**2011**)**

  Find other constructions, in particular ones that produce groups of higher class.

- **Problem 2**
  What about groups with exactly two conjugacy class sizes?

Let's go in history

- A finite group G is said to be of *conjugate type* $(1 = m_0, m_1, \ldots, m_r)$; if m_i's are precisely the different sizes of conjugacy classes of G. Here we say that G is of conjugate rank r.

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- In a series of paper *"On finite groups with given conjugate type I, II, III (1953, 1970, 1970)"*, he studied finite groups with $2, 3, 4$ conjugacy class sizes respectively.
- In this talk, we concentrate mainly on finite groups with exactly two conjugacy class sizes.

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- $G = P \times A$, where P is the non-abelian sylow p-subgroup of G and A is an abelian p'-subgroup of G.
- In particular, G is nilpotent.

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Classify finite p-groups of conjugate type $(1, p^n)$, where $n \geq 1$.

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Let G be a finite p-group with conjugate type $(1, p^n)$. Then $\exp(G/Z(G)) = p$.

Corollary 2

Let G be a finite 2-group with conjugate type $(1, 2^n)$. Then nilpotency class of G is exactly 2.

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Mann and Isaacs independently generalized this.

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Now, we concentrate on Problem 4;

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1. class 2 and conjugate type $(1, p^n)$; for $n \leq 3$.
2. class 3 and conjugate type $(1, p^n)$; for all $n \geq 1$.

Isoclinism(P.Hall,1940)

Two finite groups G and H are called *isoclinic* if there exists an isomorphism ϕ of the factor group $\bar{G} = G/\mathbb{Z}(G)$ onto $\bar{H} = H/\mathbb{Z}(H)$, and an isomorphism θ of the subgroup G' onto H' such that the following diagram is commutative

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Before going to the classification, we exhibit some examples.

- For any positive integer $r \geq 1$ and prime $p > 2$, consider the following group constructed by N. Ito.

$$\begin{aligned} G_r &= \big\langle a_1, \ldots, a_{r+1} \mid [a_i, a_j] = b_{ij}, [a_k, b_{ij}] = 1, \\ &\qquad a_i^p = a_{r+1}^p = b_{ij}^p = 1, 1 \leq i < j \leq r+1, 1 \leq k \leq r+1 \big\rangle. \end{aligned}$$

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- The family of Camina special p-groups G, with $|G'| = p^k$ provides a huge source of examples of groups of conjugate type $(1, p^k)$ and class 2.

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- Only examples known are for p-group of conjugate type $(1, p^n)$ and nilpotency class 3, where n is even integer.
- These examples were appeared in the construction of certain Camina p-groups of class 3 by Dark and Scoppola in 1996.
- It can be showed that; for fix n, the p-group of conjugate type $(1, p^{2n})$ and class 3, constructed by Dark and Scoppola is isomorphic to $\mathcal{H}_n/Z(\mathcal{H}_n)$,where $\mathcal{H}_n$ can be presented as below;

$$\mathcal{H}_n = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 \\ c & b & 1 & 0 & 0 \\ d & ab-c & a & 1 & 0 \\ f & e & c & b & 1 \end{bmatrix} : a, b, c, d, e, f \in \mathbb{F}_{p^n} \right\}.$$

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Theorem(K.Ishikawa, 1999)
A finite p-group G has exactly two conjugacy class sizes 1 and p if and only if G is isoclinic to an **extra special** p**-Group**.

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  2. G_r, for $r = 2$.

$$\begin{aligned} G_2 &= \langle a_1, a_2, a_3 \mid [a_i, a_j] = b_{ij}, [a_k, b_{ij}] = 1, \\ &\qquad a_i^p = a_3^p = b_{ij}^p = 1, 1 \leq i < j \leq 3, 1 \leq k \leq 3 \rangle. \end{aligned}$$

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- Let G be a finite p-group of conjugate type $(1, p^2)$ and nilpotency class 3. Then G is isoclinic to W, where W can be presented as,

$$\begin{aligned} W &= \langle a_1, a_2 \mid [a_1, a_2] = b, [a_i, b] = c_i, \\ &\qquad a_i^p = b^p = c_i^p = 1, i = 1, 2 \rangle. \end{aligned}$$

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Note that W is isomorphic to the the group constructed by Dark and Scoppola,$\mathcal{H}_n/Z(\mathcal{H}_n)$; for $n = 1$.

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- A finite Camina p-group of nilpotency class 2 with commutator subgroup of order p^3;

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$$\begin{aligned} G_3 &= \Big\langle a_1, \ldots, a_4 \mid [a_i, a_j] = b_{ij}, [a_k, b_{ij}] = 1, \\ &\qquad a_i^p = a_4^p = b_{ij}^p = 1, 1 \le i < j \le 4, 1 \le k \le 4 \Big\rangle. \end{aligned}$$

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- The quotient group G_3/M, where M is a normal subgroup of G_3 given by $M = \langle [a_1, a_2][a_3, a_4] \rangle$;

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- The quotient group G_3/N, where N is a normal subgroup of G_3 given by $N = \langle [a_1,\ a_2][a_3,\ a_4],\ [a_1,\ a_3][a_2,\ a_4]^t \rangle$, with t any fixed integer non-square modulo p.

- Let $\hat{G}_n$ denote the family consisting of $(n+1)$-generator non-abelian special p-groups G of order $p^{(n+1)(n+2)/2}$. Let $\hat{\mathcal{G}}_3$ denote the subfamily of $\hat{G}_3$ consisting of 2-groups of exponent 4.

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- Then it follows that all groups of this family are of conjugate type $\{1, 2^3\}$. It also turns out that any two groups in $\hat{\mathcal{G}}_3$ are isoclinic.
- For simplicity of notation, we assume that a group $\mathcal{G}$ from $\hat{\mathcal{G}}_3$ is minimally generated by the set $\{a, b, c, d\}$.

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- **Question 6**
  Does there exist a finite p-group of nilpotency class 3 and conjugate type $\{1, p^n\}$, for odd prime p and odd integer $n \geq 5$?

Let's summerize the results on finite p-groups of conjugate type $(1, p^n)$.

- For $p = 2$, such groups have nilpotency class exactly 2.
- For odd prime p, such groups can have nilpotency class 2 or 3.
- For class 2, there are many known examples.
- For class 3, there does not exist any p-group of conjugate type $(1, p^n)$, when $n = 1$ or 3.
- For class 3, there is only one example known, that too when n is even.

All these information lead to the following natural questions.

- **Question 6**

  Does there exist a finite p-group of nilpotency class 3 and conjugate type $\{1, p^n\}$, for odd prime p and odd integer $n \geq 5$?
- **Question 7**

  For given even integer n, does there exist more groups of conjugate type $(1, p^n)$, other than the example constucted by Dark and Scoppola ?

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- There exists a unique (up to isoclinism) p-group of conjugate type $(1, p^{2n})$ and class 3.
- In particular, it is isoclinic to the group constructed by Dark and Scoppola.

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Thank You