Groups where the twisted conjugacy class of the unit element is a subgroup

Timur Nasybullov

KU Leuven, Campus KULAK

Groups St Andrews, Birmingham, August 6, 2017

Daciberg Lima Gonçalves (USP)

Definitions

Let G be a group, φ be an automorphism of G.

Definitions

Let G be a group, φ be an automorphism of G.

Definition

Two elements $x, y \in G$ are called (twisted) φ-conjugated if there exists an element $z \in G$ such that

$$x = zy\varphi(z)^{-1}.$$

Definitions

Let G be a group, φ be an automorphism of G.

Definition

Two elements $x, y \in G$ are called (twisted) φ-conjugated if there exists an element $z \in G$ such that

$$x = zy\varphi(z)^{-1}.$$

$[x]_\varphi$ – φ-conjugacy class of the element x.

Definitions

Let G be a group, φ be an automorphism of G.

Definition

Two elements $x, y \in G$ are called (twisted) φ-conjugated if there exists an element $z \in G$ such that

$$x = zy\varphi(z)^{-1}.$$

$[x]_\varphi$ – φ-conjugacy class of the element x.

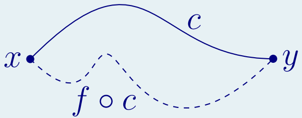
$R(\varphi)$ – number of φ-conjugacy classes (Reidemeister number).

Fixed point theory

Let X be a finite polyhedron, $f : X \to X$ be a homeomorphism.

Definition

Two points $x, y \in \mathrm{Fix}(f)$ belong to the same fixed point class of f if there exists a path c connecting x and y such that $c \simeq f \circ c$.

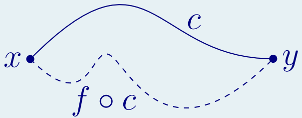


Fixed point theory

Let X be a finite polyhedron, $f : X \to X$ be a homeomorphism.

Definition

Two points $x, y \in \mathrm{Fix}(f)$ belong to the same fixed point class of f if there exists a path c connecting x and y such that $c \simeq f \circ c$.



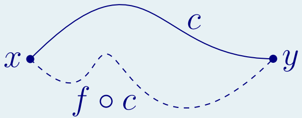
$R(f)$ – number of fixed point classes of f.

Fixed point theory

Let X be a finite polyhedron, $f : X \to X$ be a homeomorphism.

Definition

Two points $x, y \in \mathrm{Fix}(f)$ belong to the same fixed point class of f if there exists a path c connecting x and y such that $c \simeq f \circ c$.



$R(f)$ – number of fixed point classes of f.

Denote by φ the automorphism of $\pi_1(X)$ induced by f.

Then $R(f) = R(\varphi)$.

Group theory

Group theory

Conjecture (Fel'shtyn-Troitsky, 2012)

Let G be a finitely generated residually finite group. If G possesses an automorphism φ with $R(\varphi) < \infty$, then G is almost solvable.

Group theory

Conjecture (Fel'shtyn-Troitsky, 2012)

Let G be a finitely generated residually finite group. If G possesses an automorphism φ with $R(\varphi) < \infty$, then G is almost solvable.

This conjecture is known to be correct when

- φ has prime order (Jabara, 2008).
- G is linear (Fel'shtyn-N., 2016).
- other very specific conditions hold (Fel'shtyn-N., 2016).

Question

$[e]_\varphi = \{x^{-1}\varphi(x) \mid x \in G\}$ – φ-conjugacy class of the unit element.

Question

$[e]_\varphi = \{x^{-1}\varphi(x) \mid x \in G\}$ – φ-conjugacy class of the unit element.

Proposition (Fel'shtyn-Troitsky)

The twisted conjugacy class $[e]_\varphi$ of the unit element e is a subgroup of an abelian group G. The other ones are cosets $[x]_\varphi = x[e]_\varphi$.

KU LEUVEN kulak

Question

$[e]_\varphi = \{x^{-1}\varphi(x) \mid x \in G\}$ – φ-conjugacy class of the unit element.

Proposition (Fel'shtyn-Troitsky)

The twisted conjugacy class $[e]_\varphi$ of the unit element e is a subgroup of an abelian group G. The other ones are cosets $[x]_\varphi = x[e]_\varphi$.

Question

For which groups the twisted conjugacy class of the unit element is a subgroup for every automorphism?

Question

$[e]_\varphi = \{x^{-1}\varphi(x) \mid x \in G\}$ – φ-conjugacy class of the unit element.

Proposition (Fel'shtyn-Troitsky)

The twisted conjugacy class $[e]_\varphi$ of the unit element e is a subgroup of an abelian group G. The other ones are cosets $[x]_\varphi = x[e]_\varphi$.

Question

For which groups the twisted conjugacy class of the unit element is a subgroup for every automorphism?

Is it true that such group must be abelian?

Conjecture

Conjecture (Bardakov-N.-Neshchadim, 2013)

If in group G the twisted conjugacy class of the unit element is a subgroup for every automorphism, then this group is nilpotent.

Kourovka notebook, Problem 18.14, 2014.

Known result

Let $g \in G$ and $\varphi : x \mapsto x^g = g^{-1}xg$ be an inner automorphism.
Denote the class $[e]_\varphi$ by $[e]_g$.

Known result

Let $g \in G$ and $\varphi : x \mapsto x^g = g^{-1}xg$ be an inner automorphism.
Denote the class $[e]_\varphi$ by $[e]_g$.

Theorem (Bardakov-N.-Neshchadim, 2013)

Let G be a group such that the class $[e]_g$ is a subgroup of G for every $g \in G$. If G satisifies both descending and ascending chain conditions for normal subgroups, then G is nilpotent.

Known result

Let $g \in G$ and $\varphi : x \mapsto x^g = g^{-1}xg$ be an inner automorphism. Denote the class $[e]_\varphi$ by $[e]_g$.

Theorem (Bardakov-N.-Neshchadim, 2013)

Let G be a group such that the class $[e]_g$ is a subgroup of G for every $g \in G$. If G satisifies both descending and ascending chain conditions for normal subgroups, then G is nilpotent.

If G is finite, then G is nilpotent.

Example

Is it possible to prove the conjecture using only inner automorphisms?

Example

Is it possible to prove the conjecture using only inner automorphisms?

Let $G_n = \mathbb{Z}_{3^n} \rtimes \mathbb{Z}_{3^{n-1}} = \langle x, y \mid x^{3^n} = 1, y^{3^{n-1}} = 1, x^y = x^4 \rangle$.

Example

Is it possible to prove the conjecture using only inner automorphisms?

Let $G_n = \mathbb{Z}_{3^n} \rtimes \mathbb{Z}_{3^{n-1}} = \langle x, y \mid x^{3^n} = 1, y^{3^{n-1}} = 1, x^y = x^4 \rangle$.

$G = \prod_n G_n$ is not nilpotent.

Example

Is it possible to prove the conjecture using only inner automorphisms?

Let $G_n = \mathbb{Z}_{3^n} \rtimes \mathbb{Z}_{3^{n-1}} = \langle x, y \mid x^{3^n} = 1, y^{3^{n-1}} = 1, x^y = x^4 \rangle$.

$G = \prod_n G_n$ is not nilpotent.

Question

Let G be a group such that $[e]_g$ is a subgroup of G for every $g \in G$. Is it true that G is residually nilpotent?

New result

$$[e]_{g_1} > [e]_{[g_1,g_2]} > [e]_{[g_1,g_2,g_3]} > \dots$$

New result

$$[e]_{g_1} > [e]_{[g_1,g_2]} > [e]_{[g_1,g_2,g_3]} > \dots$$

Question

Let G be a finitely generated group such that $[e]_g \leq G$ for every $g \in G$. Is it true that $\cap_n [e]_{[g_1,\dots,g_n]} = \{e\}$ for every $g_1, g_2, \dots$?

New result

$$[e]_{g_1} > [e]_{[g_1,g_2]} > [e]_{[g_1,g_2,g_3]} > \dots$$

Question

Let G be a finitely generated group such that $[e]_g \leq G$ for every $g \in G$. Is it true that $\cap_n [e]_{[g_1,\dots,g_n]} = \{e\}$ for every $g_1, g_2, \dots$?

Theorem (Gonçalves-N.,2017)

Let G be a finitely generated group such that $[e]_g \leq G$ for every g. If $\cap_n [e]_{[g_1,\dots,g_n]} = \{e\}$ for every sequence $g_1, g_2, \dots,$ then either $\gamma_n(G) = \{e\}$ or $\gamma_n(G) \neq \gamma_{n+1}(G)$.

New result

$$[e]_{g_1} > [e]_{[g_1,g_2]} > [e]_{[g_1,g_2,g_3]} > \dots$$

Question

Let G be a finitely generated group such that $[e]_g \leq G$ for every $g \in G$. Is it true that $\cap_n [e]_{[g_1,\dots,g_n]} = \{e\}$ for every $g_1, g_2, \dots$?

Theorem (Gonçalves-N.,2017)

Let G be a finitely generated group such that $[e]_g \leq G$ for every g. If $\cap_n [e]_{[g_1,\dots,g_n]} = \{e\}$ for every sequence $g_1, g_2, \dots$, then either $\gamma_n(G) = \{e\}$ or $\gamma_n(G) \neq \gamma_{n+1}(G)$.

If G satisfies the descending chain condition, then G is nilpotent.

Verbal width

Let $w(x_1, \ldots, x_n) \in F_n = \langle x_1, \ldots x_n \rangle$.

Definition

The group $w(G) = \langle w(g_1, \ldots, g_n) \mid g_1, \ldots, g_n \in G \rangle$ is called the verbal subgroup of G defined by the word $w \in F_n$.

Verbal width

Let $w(x_1, \ldots, x_n) \in F_n = \langle x_1, \ldots x_n \rangle$.

Definition

The group $w(G) = \langle w(g_1, \ldots, g_n) \mid g_1, \ldots, g_n \in G \rangle$ is called the verbal subgroup of G defined by the word $w \in F_n$.

If $w = [x_1, \ldots, x_n]$, then $w(G) = \gamma_n(G)$.

Verbal width

Let $w(x_1, \ldots, x_n) \in F_n = \langle x_1, \ldots x_n \rangle$.

Definition

The group $w(G) = \langle w(g_1, \ldots, g_n) \mid g_1, \ldots, g_n \in G \rangle$ is called the verbal subgroup of G defined by the word $w \in F_n$.

If $w = [x_1, \ldots, x_n]$, then $w(G) = \gamma_n(G)$.

For $g \in w(G)$ let $l_w(g) = \min\{k \mid g = \prod_{j=1}^{k} w(g_{1i}, \ldots, g_{ni})^{\epsilon_i}\}$.

Verbal width

Let $w(x_1, \ldots, x_n) \in F_n = \langle x_1, \ldots x_n \rangle$.

Definition

The group $w(G) = \langle w(g_1, \ldots, g_n) \mid g_1, \ldots, g_n \in G \rangle$ is called the verbal subgroup of G defined by the word $w \in F_n$.

If $w = [x_1, \ldots, x_n]$, then $w(G) = \gamma_n(G)$.

For $g \in w(G)$ let $l_w(g) = \min\{k \mid g = \prod_{j=1}^{k} w(g_{1i}, \ldots, g_{ni})^{\epsilon_i}\}$.

Definition

The (verbal) width of the verbal subgroup $w(G)$ of a group G is the value $\mathrm{wid}(w(G)) = \sup\{l_w(g) \mid g \in w(G)\}$.

Verbal width

If $w = [x_1, \ldots, x_n]$, then $w(G) = \gamma_n(G)$.

Denote by $\gamma_n(x_1, \ldots, x_n) = [x_1, \ldots, x_n]$.

Verbal width

If $w = [x_1, \ldots, x_n]$, then $w(G) = \gamma_n(G)$.

Denote by $\gamma_n(x_1, \ldots, x_n) = [x_1, \ldots, x_n]$.

Theorem (Gonçalves-N., 2017)

Let G be a group with n generators such that $[e]_g \leq G$ for every $g \in G$. Then

1. $\mathrm{wid}(\gamma_2(G)) \leq n - 1$,
2. $\mathrm{wid}(\gamma_k(G)) \leq n^{k-2}(n-1)/2$ for $k \geq 3$.

Verbal width

If $w = [x_1, \ldots, x_n]$, then $w(G) = \gamma_n(G)$.

Denote by $\gamma_n(x_1, \ldots, x_n) = [x_1, \ldots, x_n]$.

Theorem (Gonçalves-N., 2017)

Let G be a group with n generators such that $[e]_g \leq G$ for every $g \in G$. Then

1. $\mathrm{wid}(\gamma_2(G)) \leq n - 1$,
2. $\mathrm{wid}(\gamma_k(G)) \leq n^{k-2}(n-1)/2$ for $k \geq 3$.

If $n = 2$, then $\mathrm{wid}(\gamma_2(G)) \leq 1$, $\mathrm{wid}(\gamma_3(G)) \leq 1$.

Verbal width

Question

Is it possible to improve the estimation of $\mathrm{wid}(\gamma_k(G))$ for $k > 3$?

Verbal width

Question

Is it possible to improve the estimation of $\mathrm{wid}(\gamma_k(G))$ for $k > 3$?

Proposition (Gonçalves-N., 2017)

Let G be a metabelian group with n generators such that $[e]_g \leq G$ for every $g \in G$. Then for $k > 1$

$$\mathrm{wid}(\gamma_{k+2}(G)) \leq \frac{n(n-1)}{2}\binom{n+k-2}{k-1}.$$

KU LEUVEN kulak

Open problems

1. Let G be a finitely generated group such that $[e]_g \leq G$ for every $g \in G$. Is it true that G is residually nilpotent?
2. Let G be a finitely generate group such that $[e]_g \leq G$ for every $g \in G$. Is it true that $\cap_n [e]_{[g_1,\ldots,g_n]} = \{e\}$ for every sequence $g_1, g_2, \ldots$?
3. Let G be a finitely generated group such that $[e]_g \leq G$ for every $g \in G$. Find a sharp estimation of $\mathrm{wid}(\gamma_k(G))$ for $k > 3$?