On the Covering Numbers of Small Symmetric and Alternating Groups, and Some Sporadic Groups

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Abstract

- We say that a group *G* has a finite covering if *G* is a set theoretical union of finitely many proper subgroups. By a result of B.Neumann this is true iff the group has a finite non-cyclic homomorphic image. Thus, it suffices to restrict our attention to finite groups. The minimal number of subgroups needed for such a covering is called the covering number of *G* denoted by б(*G*).

- Let S_n be the symmetric group on *n* letters. For odd *n* Maroti determined б(S_n)= 2^{n-1} with the exception of *n* = 9, and gave estimates for *n* even showing that б(S_n) ≤ 2^{n-2}. Using *GAP* calculations, as well as incidence matrices and linear programming, we show that б(S_8) = 64, б(S_{10}) = 221, б(S_{12}) = 761. We also show that Maroti 's result for odd *n* holds without exception proving that б(S_9)=256

- We establish in addition that, the *Mathieu* group m_{12} has covering number 208, and improve the estimate for the *Janko* group J_1 given by P.E. Holmes. (L-C K., D.N., E.S.)

- We also determine б(A_9)=157, б(A_{11})=2751 (S.M., D.N., M.E.)

The Covering Number

- **Theorem 1 (Tomkinson,1997):** *Let G be a finite soluble group and let p^α be the order of the smallest chief factor having more that one complement. Then* $\boldsymbol{\sigma(G) = p^\alpha + 1}$.

- The author suggested the investigation of the covering number of simple groups.

Linear Groups

- **Theorem 2 (Bryce, Fedri, Serena, 1999)**
- *$\sigma(G)=1/2\ q(q+1)$ when q is even,*
- *$\sigma(G)=1/2\ q(q+1)+1$ when q is odd,*

where $G=PSL(2,q)$, $PGL(2,q)$, or $GL(2,q)$, and $q \neq 2, 5, 7, 9$.

Suzuki Groups

- **Theorem 3 (Lucido, 2001)**
- *$\sigma(Sz(q)) = \frac{1}{2}\, q^2(q^2+1)$, where $q = 2^{2m+1}$.*

Sporadic Simple Groups

Theorem 6 (P.E. Holmes, 2006)

$\sigma(m_{11})=23$, $\sigma(m_{22})= 771$, $\sigma(m_{23})=41079$,

$\sigma(Ly) = 112845655268156$,

$\sigma(O'N) = 36450855$

$5165 \leq \sigma(J_1) \leq 5415$

$24541 \leq \sigma(McL) \leq 24553$.

The author has used *GAP*, the *ATLAS*, and Graph Theory.

Symmetric and Alternating Groups

- **Theorem 4** (Maroti, 2005)
- $\sigma(S_n) = 2^{n-1}$ *if n is odd,* $n \neq 9$
- $\sigma(S_n) \leq 2^{n-2}$ *if n is even.*
- $\sigma(A_n) \geq 2^{n-2}$ *if* $n \neq 7,9$, *and* $\sigma(A_n) = 2^{n-2}$ *if n is even but not divisible by 4.*
- $\sigma(A_7) \leq 31$, *and* $\sigma(A_9) \geq 80$.

Alternating Groups

- **Theorem 5 (Luise-Charlotte Kappe, Joanne Redden, 2009)**
- $\sigma(A_7) = 31$
- $\sigma(A_8) = 71$
- $127 \leq \sigma(A_9) \leq 157$
- $\sigma(A_{10}) = 256$.

Recent results

- We can now prove the exact numbers:
- **$\sigma(S_8) = 64$**
- **$\sigma(S_9) = 256$**
- **$\sigma(S_{10})=221$**
- **$\sigma(S_{12})=761$**
- **$\sigma(A_9) = 157$** *(M.E., S.M., D.N.)*
- **$\sigma(A_{11}) = 2751$** *(M.E.,S.M.,D.N.)*
- **$5316 \leq \sigma(J_1) \leq 5413$.**

Starting point

- It is sufficient to consider the number of maximal subgroups of *G* needed to cover all maximal cyclic subgroups of *G*.

- We used *GAP* for the distribution of the elements in the maximal subgroups

- We first estimated the limits by a *Greedy Algorithm.*

Note:

- **Easy case**: When the elements are partitioned into the subgroups of a conjugacy class.
- **Harder case**: When the elements of a certain cyclic structure are not partitioned.
- **Further Approaches:**

➢ Incidence matrices and Combinatorics

➢ Linear programming

S7

Maximal subgroups	Order of Class Representative	Size
MS1 = A_7	2520	**1**
MS2 = S_6	720	**7**
MS3 = S_3 x S_4	144	**35**
MS4 = C_2 x S_5	240	**21**
MS5 = (C_7:C_3):C_2	**42**	**120**

Distribution of the Elements of S7:

Order	Cyclic Structure	Size	MS1=A_7	MS2	MS3	MS4	MS5
1	1	1	1				
2	(12)	21	0	15	9	11	**0**
2	(12)(34)		X				
2	(12)(34)(56)	105	0	15,P	9	15	**7**
3	(123)		X				
3	(123)(456)		X				
4	(1234)	210	0	90	6,P	30	**0**
4	(1234)(56)		X				
5	(12345)		X				
6	(123456)	840	0	120,P	0	0	**14**
6	(123)(45)	420	0	120	36	40	**0**
6	(123)(45)(67)		X				
7	(1234567)	720	X				
10	(12345)(67)	504	0	0	0	24,P	**0**
12	**(1234)(567)**	**420**	**0**	**0**	**12,P**	**0**	**0**

S7

- It is clear from the table why $\sigma = 2^{7-1}$
- The group is covered by A_7 (MS1), the 7 groups S_6 in MS2, the 35 groups in MS3, and the 21 groups in MS4: $1+7+35+21=64=2^6$.

- $\sigma(S7) = 64.$

S8

Maximal subgroups	**Order of Class Representative**	**Size**
MS1 = A_8	20160	**1**
MS2 = S_3 x S_5	720	**56**
MS3 = C_2 x S_6	1440	**28**
MS4 = S_7	5040	**8**
MS5=((((C_2xD_8):C_2):C_3):C_2):C_2	384	**105**
MS6 = (S_4 x S_4): C_2	1152	**35**
MS7 = PSL(3,2): C_2	**336**	**120**

S8

Distribution of Elements:

Order	Cyclic Structure	Size	MS1	MS2	MS3	MS4	MS5	MS6	MS7
1	1	1	1	1	1	1	1	1	**1**
2	2^1	28	0	13(26)	16(16)	21(6)	4(15)	12(15)	**0**
2	2^2	210	210, P	45(12)	60(8)	105(4)	18(9)	42(7)	**0**
2	2^3	420	0	45(6)	60(4)	105(2)	28(7)	36(3)	**28(8)**
2	2^4	105	105, P	0	15(4)	0	25(25)	33(11)	**21(24)**
3	3^1	112	112, P	22(11)	40(10)	70(5)	0	16(5)	**0**
4	2x4	2520	2520,P	90(2)	180(2)	630(2)	24,P	72,P	**0**
4	4^1	420	0	30(4)	90(6)	210(4)	12(3)	12,P	**0**
4	2^2x 4	1260	0	0	90(2)	0	36(3)	180(5)	**0**
4	4^2	1260	1260,P	0	0	0	60(5)	108(3)	**42(4)**
5	5	1344	1344,P	24,P	144(3)	504(3)	0	0	**0**
6	2x3	1120	0	100(5)	160(4)	420(3)	0	96(3)	**0**
6	2x2x3	1680	1680,P	90(3)	120(2)	210,P	0	48,P	**0**
6	2x3^2	1120	0	40(2)	40,P	0	32(3)	0	**0**
6	6	3360	0	0	120,P	840(2)	32,P	0	**56(2)**
6	2 x 6	3360	3360,P	0	120,P	0	32,P	192(2)	**0**
7	7	5760	5760,P	0	0	720,P	0	0	**48,P**
8	8	5040	0	0	0	0	48,P	144,P	**84(2)**
10	2 x 5	4032	0	72,P	144,P	504,P	0	0	**0**
12	3 x 4	3360	0	60,P	0	420,P	0	96,P	**0**
15	**3 x 5**	**2688**	**2688,P**	**48,P**	**0**	**0**	**0**	**0**	**0**

S8

- Here are the maximal subgroups and the distribution of the elements of S_8 in the representatives of the maximal subgroups. In parentheses the small numbers mean in how many representatives each element is to be found.
- Example: Each element of order 6 of type 2x3 i.e. (1,2)(3,4,5) is to be found in 3 representatives of MS4, and in each representative of MS4 there are 420 such elements.
- The group is covered by A_8 (MS1), the 28 groups in MS3, and the 35 groups in MS6, i.e. $1+28+35 = 64 = 2^6$.
- $\sigma(S8) = 64$.
- **The difficulty consists to prove that this is a minimal covering.**
- We first did that computationally using *GAP* and *Gurobi* optimizer that we confirm later by a paper proof.

The Covering Number of S_{10}

- To determine a minimal covering by maximal subgroups, it suffices to find a minimal covering of the conjugacy classes of maximal cyclic subgroups by maximal subgroups of the group.

Maximal subgroups

Maximal subgroups (3977)	**Order of Class Representative**	**Size**
MS1 = A_10	1814400	**1**
MS2=S_4 x S_6	17280	**210**
MS3 = S_3 x S_7	30240	**120**
MS4 = C_2 x S_8	80640	**45**
MS5 = S_9	362880	**10**
MS6= C_2 x (((C_2xC_2xC_2xC_2):A_5):C_2	3840	**945**
MS7 = (S_5 x S_5):C_2	28800	**126**
MS8 = (A_6.C_2):C_2	**1440**	**2520**

Distribution of elements generating maximal cyclic subgroups:

Order	Cyclic Structure	Size	MS1	MS2	MS3	MS4	MS5	MS6	MS7	MS8
ODD										
4	2^2x 4	56700	0	1080_4	1890_4	3780_3	11340_2	180_3	900_2	**0**
4	2x4^2	56700	0	540_2	0	1260,P	0	300_5	1800_4	**90_4**
6	2^3x3	25200	0	480_4	840_4	1680_3	2520, P	0	600_3	**0**
6	2x3^2	50400	0	1200_5	1680_4	2240_2	10080_2	160_3	800_2	**0**
6	2^2x6	75600	0	360,P	0	3360_2	0	240_3	2400_4	**0**
6	3x6	201600	0	960,P	1680,P	0	20160,P	0	0	**240_3**
8	8	226800	0	0	0	5040,P	45360	240,P	0	**180**
	10	362880	0	0	0	0	0	384,P	2880,P	**144,P**
12	$3^2$4	50400	0	240,P	840_2	0	0	160_3	0	0
14	2x7	259200	0	0	2160,P	5760,P	25920,P	0	0	**0**
20	4x5	181440	0	964,P	0	0	18144,P	0	1440,P	**0**
30	2x3x5	120960	0	0	1008,P	2688,P	0	0	960,P	**0**
-EVEN------	----------	---------	--------	---------	--------	---------	---------	-------	--------	**------**
6	2 x 6	151200	P	720, P	2520_2	6720_2	30240_2	160 P	0	**0**
9	9	403200	P	0	0	0	40320, P	0	0	**0**
12	4x6	151200	P	720, P	0	0	0	160, P	2400_{x2}	**0**
12	2x3x4	151200	P	1440_2	2520_2	3360 P	15120 P	0	1200 P	**0**
21	3 x 7	172800	P	0	1440 P	0	0	0	0	**0**
8	**8x2**	**226800**	**P**	**0**	**0**	**5040,P**	**0**	**240,P**	**3600_2**	**180_2**

S10

- We first found that the Covering number has **upper bound**: MS1+MS3+MS5+MS7 =1+120+10+126=257.
- However, we ran a Greedy algorithm on MS3 and found out that 84 groups only from MS3 are sufficient to cover the elements of type 3^2x4. So:
- σ ≤ 1+84+10+126=221.
- The upper bound was reduced.
-
- **The lower bound:** The elements of type 3^2 x 4 are 50400. If they were partitioned in MS3 we would have needed 50400/840 = 60.
- So, we need at least 61 from them.
- 1+61+10+126= 198.
-
- **Hence 198 ≤ σ ≤221.**

Theorem 1:

The Covering Number of S_{10} is 221.

- *Sketch of the Proof:*
- It is not difficult to see from the Inventory that the groups from MS3, MS5, and MS7 represent a covering of the odd permutations, and MS1={A_{10}} covers the even.
- We want to minimize this covering.
- The problematic elements are of structure 3x3x4, of order 12.
- The proof further involves *Incidence matrices*, and *Combinatorics.*

Incidence matrices

- Let V, and U are two collections of objects. Call the objects in V elements, and the objects in U sets.
- The incidence structure between U and V can be represented by the incidence matrix A(*aij*) of
- (V, U):
- a_{ij}= $\begin{cases} 1 \ if \ v_i \in U_j \\ 0 \ if \ v_i \neg\in U_j \end{cases}$
- Let W be a sub-collection of U. We define a vector $x(W) = (x_1, x_2, \ldots x_{|U|})^T$ as follows
- x_j= $\begin{cases} 1 \ if \ u_j \in W \\ 0 \ if \ u \neg\in W \end{cases}$
- **Let $\boldsymbol{A * x(W) = y(W) = (y_1, y, \ldots y_{|V|})^T, where \ y_i} \geq 0$.**

- If y_i=0, then $v_i \neg\in \bigcup_{u \epsilon W}$, and
- if $y_i > 0, \forall i$, then every v_i is contained in at least one member of W. **We say that *W* covers *V*.**

- **Our goal is to minimize |W|, s. t. *W* covers *V*, i.e. maximize the number of the *0*-entries in $\boldsymbol{x(W)}$.**

The elements of type 3*3*4

- There are 50,400 elements of type 3*3*4 in S_{10}. They are to be found in MS3, but are not partitioned.
- Each class of MS3 contains 840 such elements, and each element is in exactly 2 subgroups of MS3.
- Because the subgroups of MS3 are isomorphic to $S_3 x S_7$, we can label them by the letters fixed by the respective S_7, i.e.
- MS3 = {H(k_1,k_2,k_3), k_1,k_2,k_3 ϵ {0,1,2,3,4,5,6,7,8,9}, k_1<k_2<k_3}.
- So, our incidence matrix will contain 120 columns, labeled by the members of MS3.
- The rows are the maximal cyclic subgroups generated by our elements. There are 6 cyclic subgroups of order 12 in the intersection of H(i_1, i_2, i_3) and H(i_4, i_5, i_6) generated by:
- $(i_1, i_2, i_3)(i_4, i_5, i_6)(i_7, i_8, i_9, i_{10})$,
- $(i_1, i_3, i_2)(i_4, i_5, i_6)(i_7, i_8, i_9, i_{10})$,
- $(i_1, i_2, i_3)(i_4, i_5, i_6)(i_7, i_9, i_8, i_{10})$,
- $(i_1, i_3, i_2)(i_4, i_5, i_6)(i_7, i_9, i_8, i_{10})$,
- $(i_1, i_2, i_3)(i_4, i_5, i_6)(i_7, i_8, i_{10}, i_9)$,
- $(i_1, i_3, i_2)(i_4, i_5, i_6)(i_7, i_8, i_{10}, i_9)$,

- and each one of them contains 4 elements of type 3*3*4: thus the 50,400 elements of type 3*3*4 are partitioned into 50,400/24=2100 equivalence classes. Our incidence matrix will have 2100 rows.

Confirming the result of the Greedy algorithm

- We have an incidence (0 -1) matrix *A* of size 2100 x 120 with exactly 2 entries equal to 1 in each row.
- If $x(W) = (1{,}1, \dots 1)^T$, *y*(*W*)=$A * x(W) = (2{,}2, \dots, 2)^T$.
- We want to determine the maximum numbers of 0-s entries contained in a x*(W)* vector, so that the *y(W)* vector has all non-zero entries.
- **We can achieve that by removing the maximal subset$\{\boldsymbol{U_1}, \boldsymbol{U_2}, \dots \boldsymbol{U_t}\}$ of *U* with pairwise non-trivial intersection.**

Combinatorics

- **THEOREM (Erdos, Ko, Rado): The maximal number *m* of *k*-subsets $A_{1,}\ A_2, \ldots A_m$ of an *n*-set *S* that are pairwise non-disjoint is $m \leq \binom{n-1}{k-1}$.**

- **The upper bound is best possible, and it is attained when A_i are precisely those *k*-subsets of *S* which contain a chosen fixed element of *S*.**

Corollary

Proposition:

The elements of type 3*3*4 in S10 are covered by 84 groups from MS3, and this is a minimal covering. In particular,

***M = MS3\ D*, where**

$\boldsymbol{D = \{H(0, k_1, k_2); k_1, k_2} \in$ **{1, 2, …9},** $\boldsymbol{k_2 < k_3\}}$

is a minimal covering.

Proof:

According to the Theorem (*n*=10, *k*=3), the maximal subset$\{U_{1,}\ U_{2,} \ldots U_{t\}}$ of *U* with pairwise non-trivial intersection has cardinality: *m*=$\binom{9}{2}$=36. Therefore,

- 120-36=84.

Proof of Theorem 1

- We shall see that б(S_{10}) = |MS1|+|MS5|+|MS7|+84= 221 .
- The elements of order 21 are only to be found in MS1 and MS3, in both they are partitioned, so we take MS1={A10}, size 1.
- The elements of order 10 are partitioned in MS6, MS7, and MS8. MS7 has the least size: 126.
- The elements of order 14, type 2*7 are partitioned in MS3, and MS5. If H(0,k1,k2) is removed from MS3, they will no longer be covered by MS3. They can only be covered by all 10 members of MS5.
- Together with the result for the elements of type 3*3*4, we have:
- б(S_{10})= 1+ 126+ 10+ 84 = 221.

S9

Maximal subgroups (1376)	Order of Class Representative	Size
MS1 = A_9	181440	**1**
MS2 = S_4 x S_5	2880	**126**
MS3 = S_3 x S_6	4320	**84**
MS4 = C_2 x S_7	10080	**36**
MS5= S_8	40320	**9**
MS6 = ((((C_3x((C_3xC_3):C_2)):C_2):C_3):C_2):C_2	1296	**280**
MS7 = (((C_3xC_3):Q_8):C_3):C_2	**432**	**840**

S9

Distribution of Elements:

Order	Cyclic Structure	Size	MS1	MS2	MS3	MS4	MS5	MS6	MS7
1	1	1	1	1	1	1	1	1	**1**
2	2^1		0						
2	2^2								
2	2^3		0						
2	2^4								
3	3^1								
3	3^2								
3	3^3								
4	2x4	7560	7560,P						
4	4^1	756	0	36(6)	90(10)	210(10)	420(5)	0	**0**
4	2^2x 4	11340	0	180(2)	270(2)	630(2)	1260,P	162(4)	**0**
4	4^2 ~~=8^2~~								
5	5	3024	3024,P						
6	2x3	2520	0	220(11)	270(9)	490(7)	1120(4)	36(4)	**0**
6	2^2x3	7560	7560,P						
6	2x3^2	10080	0	160(2)	360(3)	280,P	1120,P	36, P	**0**
6	6	10080	0	0	120,P	840(3)	3360(3)	36, P	**56(2)**
6	2 x 6	30240	30240,P						
6	$2^3$3	2520	0	60(3)	30, P	210(3)	0	36(4)	**0**
6	3x6	20160	0	0	240,P	0	0	288(4)	**72(3)**
7	7	25920	25920,P						
8	8	45360	0	0	0	0	5040,P	0	**108(2)**
9	9	40320	40320,P						
10	2 x 5	18144	0	144,P	432(2)	1008(2)	4032(2)	0	**0**
10	$2^2$5	9072	9072,P						
12	3 x 4	15120	0	360	180,P	420,P	3360	0	**0**
14	2x7	25920	0	0	0	720,P	0	0	**0**
15	3 x 5	24192	24192,P						
20	**4x5**	**18144**	**0**	**144,P**	**0**	**0**	**0**	**0**	**0**

S9

- Here is the distribution of the elements of S_9 in the representatives of the maximal subgroups. Here is how the lower and the upper bound are clearly to be seen:
- We definitely need:
- MS1=A_9 (1 group)
- MS2 (126 groups) to cover the elements of order 20.
- MS4 (36 groups) to cover the elements of order 14, and 12.
- MS5 (9 groups) to cover the elements of order 8, and ((1,2,3)(4,5,6)(7,8).
- Then, if you take all the 84 groups of MS3, we'll cover 3 types of elements of order 6. So, 84 more groups add up to **256: the upper bound.**
- **The lower bound.:**
- If we cover the elements of type 3x6 (20160) by groups from MS6 instead (where they are not partitioned), we would have needed at least 20160/288=70 groups. So, **1+126+36+9+71=243 ≥ σ**
- **Hence, 243 ≤ σ ≤ 256.**

Linear Programming

- **Theorem**:The covering number 6(S_9) = 256.
- **Proposition:** The elements 3x6 have a minimal covering by 84 subgroups.
- **Proof: Computational** *GAP and Gurobi.*
- Using the GAP program we are setting equations readable by GUROBI. The GUROBI output shows that a minimal covering of these elements consists of 84 subgroups from MS3, MS6, and MS7. Since these elements are partitioned in MS3, these 84 subgroups constitute a minimal covering of these elements.
- The calculations were done on a Dell desktop machine with 16 GB of RAM and a Core i-7 processor. The calculation took 453 s.

J1 and KoKo

Similar approach was used for the Mathieu group M_{12} and the Janko group J_1.

- The paper can be found at: http://arxiv.org/abs/1409.2292
- However, we wanted to achieve the best possible (the smallest) range for J1 on more powerful machines. Which we did on the new super computer KoKo installed by Max Plank at the FAU Harbor branch. Here below are some characteristics of KoKo:
- 400 Intel Xeon Cores; 1000’s of Intel Xeon Phi Cores; 128GB of RAM per node; Scientific Linux 6.5; 160 terabites memory. For more details see: http://hpc.fau.edu

J1

- It was determined by Holmes that all 1540 maximal subgroups isomorphic to $C_{19}{:}C_6$ and all 2926 maximal subgroups isomorphic to S_3x$D_{10}$are needed in a minimal covering. The only remaining elements generating maximal cyclic subgroups that need to be covered are those of order 11 (type 11A), and 7 (type 7A).
- Only maximal subgroups isomorphic to PSL(2,11) are needed to cover 11A; and only $C_2^3{:}C_7{:}C_3$ are needed for type 7A.

GAP and *GUROBI*

- *GAP* is used to create a system of linear inequalities, the optimal solution to which corresponds to a minimal cover.
- *GUROBI* then performs a linear optimization on this system of linear inequalities.
- Any time the “best objective” (best actual solution) and the “best bound”(the size of the best lower bound) found by *GUROBI* get identical, *GUROBI* has found a minimal subgroup cover.

- The codes can be found at:
- http://www.math.Binghamton.edu/menger/coverings/.

J1 and Koko calculations

- The program for the elements of order 11 finished in about 2 1/2 hour. It took 196 subgroups to cover the elements of order 11 in J1.
- However, although powerful parallel computing was done on the super-computer, with optimal parameters, and using 8 nodes, it took a while to get to:
- 253860990 231372205 99% 0% 0% 752.00000 653.04421 13.2% 476 2244640s
- **Interpretation:** The lower bound we got for the elements of order 7 is 654, the upper bound – 751, and the discrepancy between the two numbers is 13.2%.
- The last calculation took 476 2244640s = 26.3 days…
- With the newest results, we can claim now that the covering number for J1 is between:
- 1540+2926+196+654=5316 and
- 1540+2926+196+751=5413, i.e.:
- **5316 ≤ $\sigma(J_1)$ ≤ 5413**
- We also tried MINION, but the problem has no better solution than the one GUROBI provided. This is as far as we can push the bounds for J1 with current techniques.

The covering number of A_9, and A_{11}.

- In another paper with Spyros Magliveras, and Michael Epstein we establish the covering number of A_9 , and A_{11} to be respectively 157, and 2751.
- For A9 we used again incidence matrices and linear programming. Problematic here were the elements of order 9 covered by 2 conjugacy classes isomorphic to *PSL2(8),* but also by another conjugacy class of maximal subgroups. Computation of the incidence matrix of order 40902 x 1615 was done by the software system KNUTH developed by SM in APL to compute with permutation groups and combinatorial objects. The large LP using GUROBI took one day to confirm the minimal covering number.

- **As of now, the smallest values of *n* for which the covering numbers of S_n, and A_n are not known are *n* = 14, and *n* = 12 respectively.**

Summary

- At this stage, we may want to raise the question whether further results on covering numbers can only be established one at a time, or if one can find methods to give general results for larger classes. Eric Swartz established some general results for larger classes of symmetric groups (having degree divisible by 6) that hold only when if *n* is sufficiently large. **It is our hope that the unknown cases can be resolved with similar techniques, leaving only a small number of cases for small values of *n* to be resolved using computation, or individual inspection.**

THANK YOU ! ☺