On primitivity of group algebras of non-noetherian groups

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1. Primitive group rings

<u>Definition (a primitive ring)</u>

Let R be a ring with the identity element,

R is right primitive $\Leftrightarrow$ $\exists M_R$ a faithful irreducible right R-module

$\Leftrightarrow$ $\exists\, \rho$: a maximal right ideal of R which contains no non-trivial ideals

- R: commutative primitive $\Rightarrow$ R is a field.
- R is simple $\Rightarrow$ R is primitive.
- R is artinian simple $\Rightarrow$ $R \simeq M_n(D) \doteq End_D(V)$, $dim_D(V) < \infty$.

$$R = End_D(V) \xrightarrow{\quad dim_D(V) = \infty \quad} R \text{ is a primitive ring.}$$

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M: a faithful right R-module :

$$r \in R;\ Mr=0 \Rightarrow r=0$$

M: an irreducible (simple) right R-module :

$$N \leq M \Rightarrow N=0 \text{ or } N=M$$

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- $G \neq 1$: finite or abelian $\Rightarrow$ KG is never primitive.

<u>For the case of noetherian groups</u>

<u>Definition (Norhterian groups)</u>

A group G is noetherian provided any subgroup of G is finitely generated.

- G is polycyclic by finite $\Rightarrow$ G is noetherian.
- G is noetherian $\Rightarrow$ G is often polycyclic by finite; it is not easy to finid noetherian but not polycyclic by finite.

- G is a polycyclic by finite group

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$\Delta(G)$: the finite conjugate center of G; $\Delta(G)=\{\, g\in G \mid [G:C_G(g)]<\infty\}$

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For the case of non-noetherian groups

If G is one of the following types of groups, then KG is primitive for any field K:

- G is a free product of non-trivial groups (except $G=\mathbb{Z}_2 * \mathbb{Z}_2$)
  →(1973, Formanek)
- G is an amalgamated free product satisfying certain conditions
  →(1989, Balogun)
- G is an ascending HNN extension of a free group →(2007, N)
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2. Main Results

We would like to determine the primitivity of group algebras of non-noetherian groups as generally as possible. To do this, we consider a condition satisfied by many class of groups.
We first explain the notations needed.

Mutually reduced sets

Let G be a group and M a subset of G.

We denote by $\widetilde{M}$ the symmetric closure of M; $\widetilde{M} = M \cup \{x^{-1} \mid x \in M\}$,

and by M^x , the set $\{x^{-1}fx \mid f \in M\}$, where $x \in G$.

For non-empty subsets $M_1, M_2, \ldots, M_n$ of G, consisting of elements $\neq 1$,

we say that $M_1, M_2, \ldots, M_n$ are mutually reduced in G, if for each finite number of elements $g_1, g_2, \ldots, g_m \in \bigcup_{i=1}^{n} \widetilde{M}_i$,

$$g_1 g_2 \cdots g_m = 1 \quad \Rightarrow \quad \exists i, j \text{ s.t. } g_i, g_{i+1} \in \widetilde{M}_j.$$

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We here consider the following condition:

$$(*)\ \begin{cases} \text{For any non-empty subsets } M \text{ of } G \text{ consisting of finite number of elements} \neq 1, \\ \text{there exist } x_1, x_2, x_3 \in G \text{ such that } M^{x_1}, M^{x_2}, M^{x_3} \text{ are mutually reduced.} \end{cases}$$

Theorem 1 ([Nishinaka and Alexander, 2017])

If G is a countable infinite group and G satisfies $(*)$,

then KG is primitive for any K.

This is true even if the cardinality of G is general

provided G has a free subgroup whose cardinality

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Most infinite groups are non-Noetherian except for polycyclic by finite groups, and they satisfy $(*)$.

For example;

a free group, a free product,

a locally free group,

an amalgamated free product,

an HNN-extension,

a one relator group with torsion ...

a non-elementary hyperbolic group
$\leftarrow$ [B. Solie, 2017, arXiv:1706.03905]

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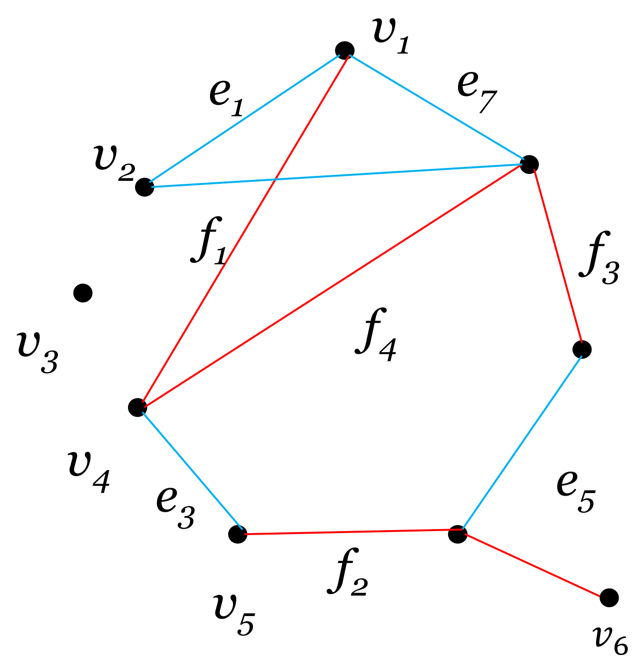
3. SR-graphs

We consider a Two-edge coloured graph which is simple graph
(an undirected graph without loops or multi-edges).

$V = \{v_1, v_2, \ldots, v_n\}$ $E = \{e_1, e_2, \ldots, e_m\}$ $F = \{f_1, f_2, \ldots, f_l\}$

An SR-graph

$\mathcal{S} = (V, E, F)$ is an SR-graph
if every component of $\mathcal{G} = (V,E)$
is a complete graph.



$I(\mathcal{G}) = \{v_3, v_6\}$

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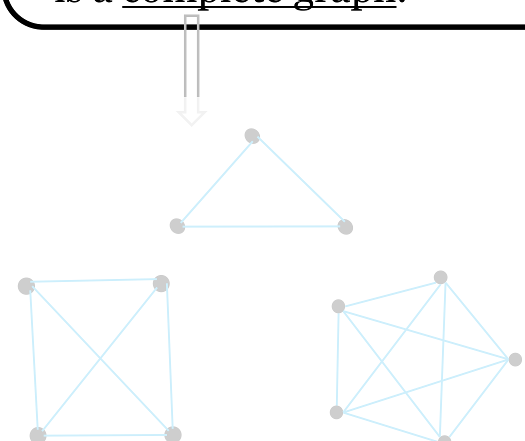
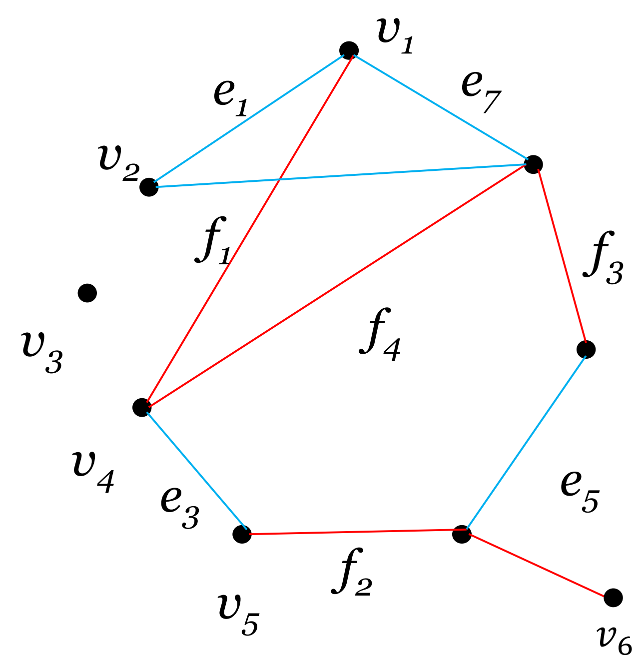
$S = (V, E, F)$ is an SR-graph if every component of $\mathcal{G} = (V,E)$ is a complete graph.





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We consider a Two-edge coloured graph which is simple graph (an undirected graph without loops or multi-edges).

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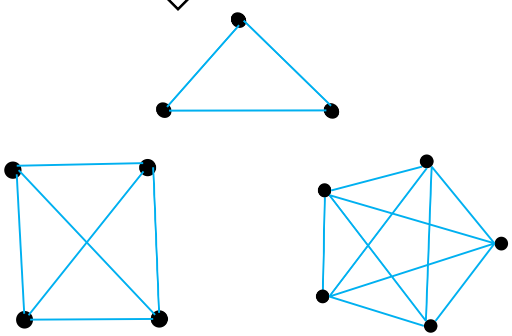

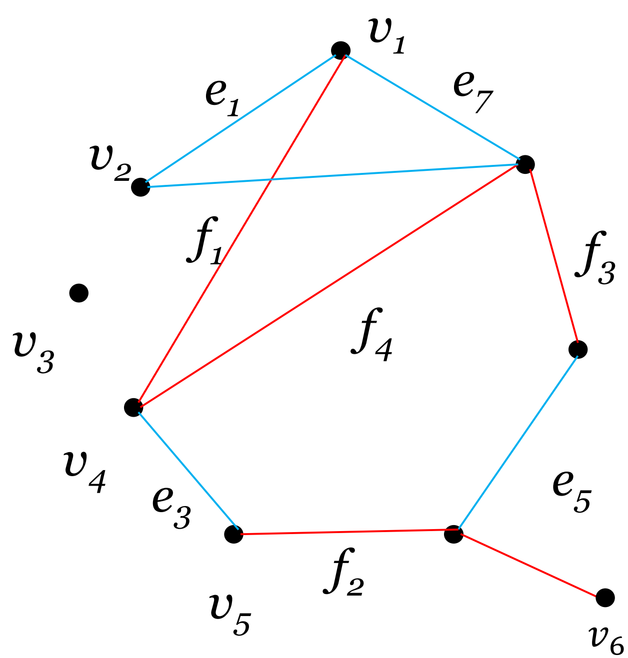

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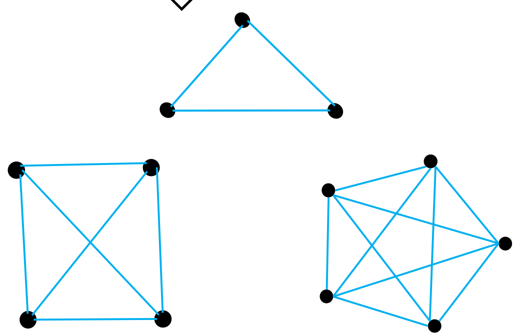

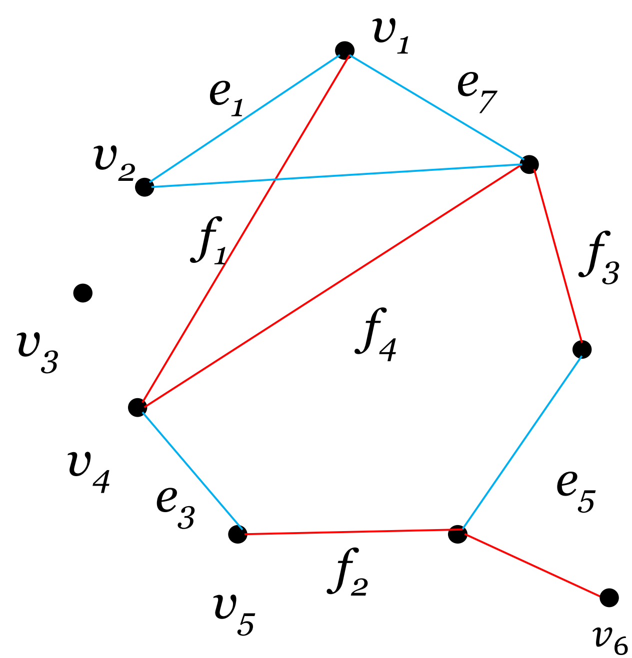

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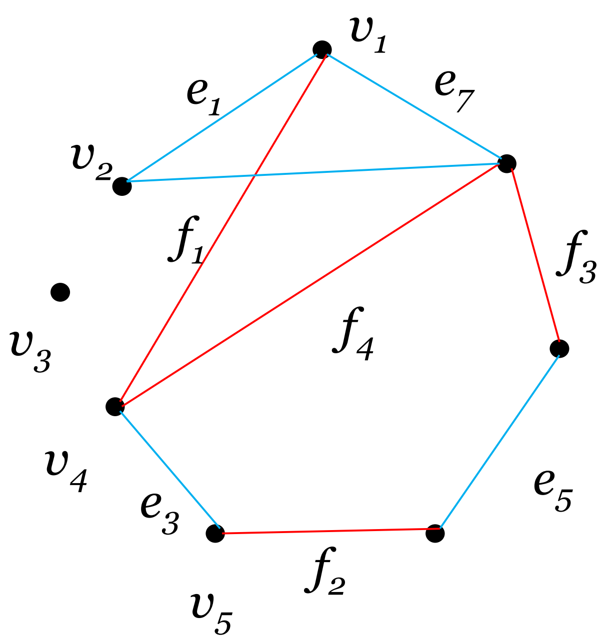
In an SR-graph, we call an alternating cycle an SR-cycle.



an SR-cycle: $f_1 e_3 f_2 e_5 f_3 e_7$

We would like to know when an SR-graph has an SR-cycle.

Results on SR-graphs

$\mathcal{S} = (V, E, F)$, $\mathcal{G} = (V, E)$, $\mathcal{H} = (V, F)$.

c($\mathcal{G}$): the number of the set of components of $\mathcal{G}$

c($\mathcal{H}$): the number of the set of components of $\mathcal{H}$

Theorem G1 ([Nishinaka and Alexander, 2017])

$\mathcal{S}$ is connected and each component of $\mathcal{H}$ is complete.

Then $\mathcal{S}$ has an SR-cycle if and only if c($\mathcal{G}$) + c($\mathcal{H}$) < $|V| + 1$.

$\mathcal{H}_i = (V_i, F_i)$ ($i=1,\ldots,n$) are the components of $\mathcal{H}$.

For $\mathcal{H}_i \cong K_{m_1,\cdots,m_t}$, let μ_i be $\max\{m_1, \cdots, m_t\}$.

$K_{2,3}$, $|V_i| < 2\mu_i$

Theorem G2 ([Nishinaka and Alexander, 2017])

Suppose that $\mathcal{H}_i$ is a complete multipartite graph for each i.

$|I(\mathcal{G})| \leq n$ and $|V_i| > 2\mu_i$ for each i $\Rightarrow$ $\mathcal{S}$ has an SR-cycle.

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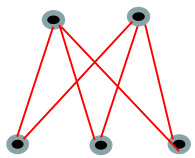

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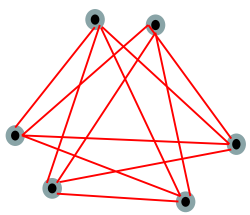

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4. An application of SR-graphs to group algebras

Let KG be the group algebra of a group G over a field K.

Let $a = \sum_{i=1}^{m} \alpha_i f_i$ and $b = \sum_{j=1}^{n} \beta_j g_j$ be in KG,

where $f_i, g_j \in G$ with $f_i \neq f_j$, $g_i \neq g_j$ $(i \neq j)$ and $\alpha_i, \beta_j \in K \setminus \{0\}$.

Suppose $ab \in K$. Then $\sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_i \beta_j f_i g_j \in K$.

If $f_i g_j \notin K$, $\exists k, l$, s.t. $f_i g_j = f_k g_l$.

Now, let $V = \{v_{ij} \mid i, j\}$ and let E be the set defined by $v_{ij} v_{kl} \in E$

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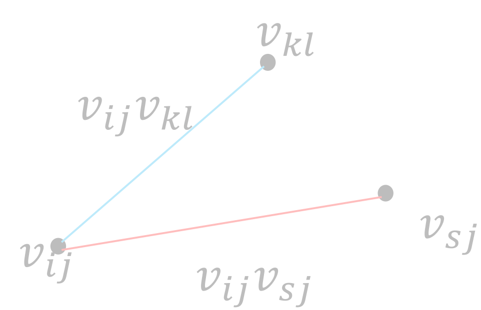

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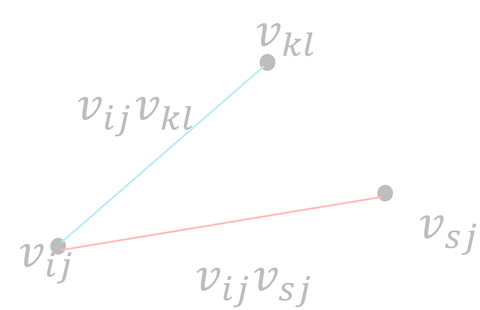

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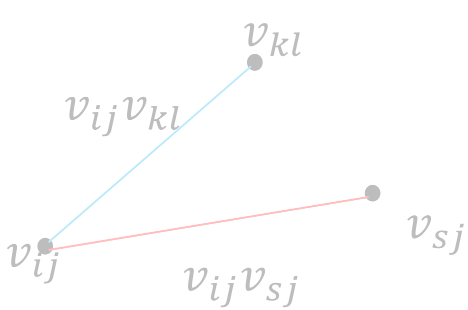

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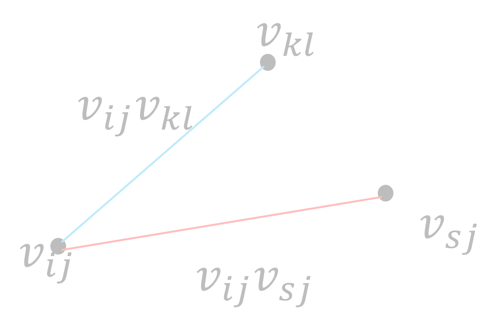

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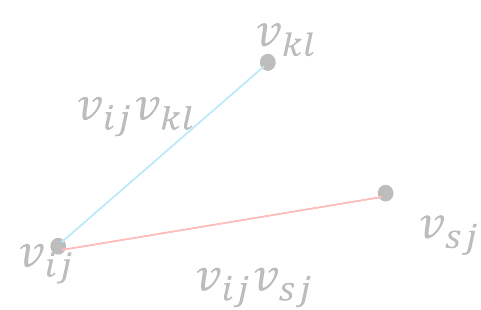

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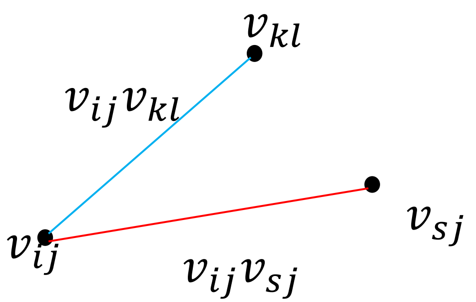

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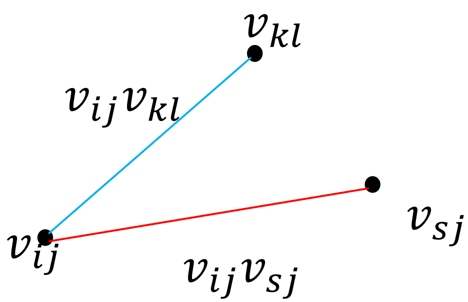

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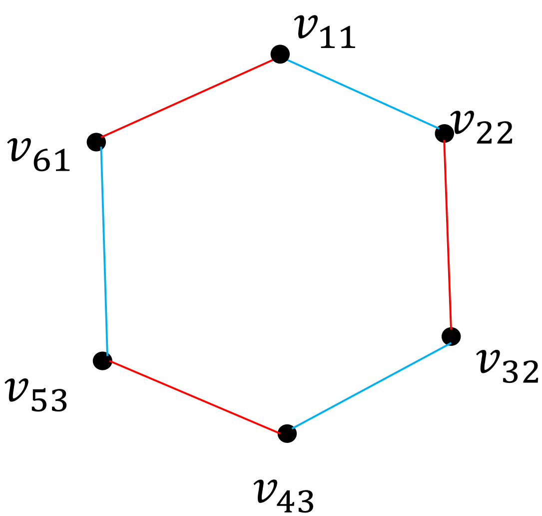

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$$f_1^{-1} f_2 f_3^{-1} f_4 f_5^{-1} f_6 = 1$$

Recall that f_i's are supports of $a = \sum_{i=1}^{m} \alpha_i f_i$. So, if we prepare f_i's so as not to satisfy the above equation, then we can conclude $ab \notin K$.

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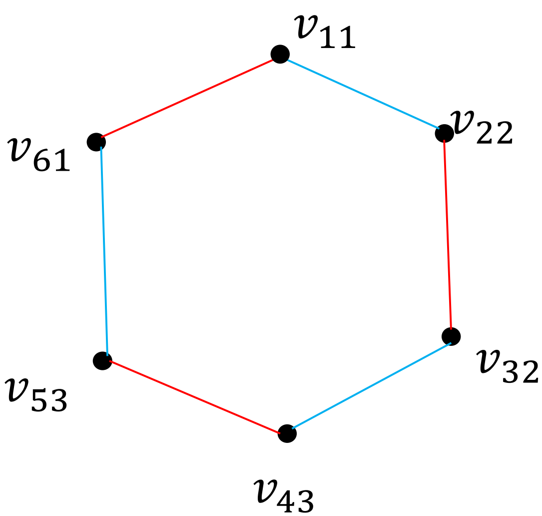

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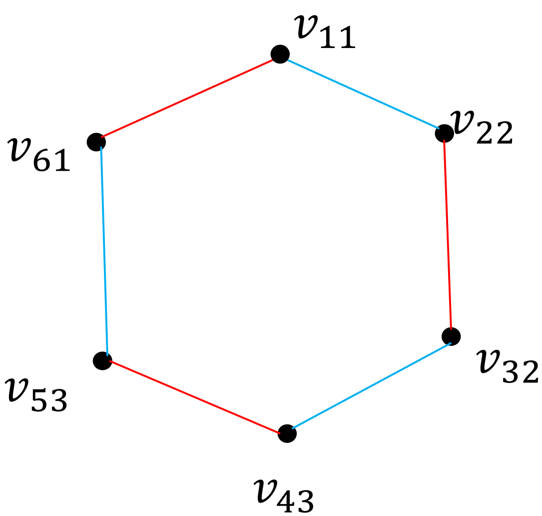

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5. How to prove primitivity of group algebras: Outline of the proof of Theorem 1

Recall:

Theorem 1 ([Nishinaka and Alexander, 2017])

> If G is a countable infinite group and G satisfies $(*)$,
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> then KG is primitive for any K.

where,

$(*)$ For any non-empty subsets M of G consisting of finite number of elements $\neq 1$, there exist $x_1, x_2, x_3 \in G$ such that $M^{x_1}, M^{x_2}, M^{x_3}$ are mutually reduced.

$$\left(g_1, g_2, \ldots, g_m \in \cup_{i=1}^{3} \widetilde{M^{x_i}},\ g_1 g_2 \cdots g_m = 1 \ \Rightarrow\ \exists i, j \text{ s.t. } g_i, g_{i+1} \in \widetilde{M^{x_j}}. \right)$$

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Formanek's Method

$a \in KG\backslash\{0\}$, $\varepsilon(a) \in KGaKG$, $\rho = \sum_{a \in KG\backslash\{0\}} (\varepsilon(a)+1)KG$.

$\rho \neq KG \Rightarrow KG$ is primitive

The main difficulty here is how to choose elements $\varepsilon(a)$'s so as to make ρ be proper.

Note that if $r \in \rho$, then $r = \sum_{t=1}^{l}(\sum_{s=1}^{3} \varepsilon(a_t) + 1)b_t$ for some a_t, b_t in KG.

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$\rho \neq KG \Rightarrow KG$ is primitive

The main difficulty here is how to choose elements $\varepsilon(a)$'s so as to make ρ be proper.

Note that if $r \in \rho$, then $r = \sum_{t=1}^{l}(\sum_{s=1}^{3} \varepsilon(a_t) + 1)b_t$ for some a_t, b_t in KG.

Let $a_t = \sum_{i=1}^{m_t} \alpha_{ti} f_{ti}$ and $b_t = \sum_{j=1}^{n_t} \beta_{tj} g_{tj}$ be in KG.

where $f_i, g_j \in G$ with $f_i \neq f_j$, $g_i \neq g_j$ $(i \neq j)$ and $\alpha_i, \beta_j \in K \setminus \{0\}$.

We can choose $\varepsilon(a_t)$ so that $r = \sum_{t=1}^{l}(\sum_{s=1}^{3} y_{ts} A_t + 1)b_t$.

where $x_{ts}, y_{ts} \in G$, $A_t = x_{t1}^{-1} a_t x_{t1} + x_{t2}^{-1} a_t x_{t2} + x_{t3}^{-1} a_t x_{t3}$.

All we have to do is to show, $r = \sum_{t=1}^{l}(\sum_{s=1}^{3} y_{ts} A_t + 1)b_t \neq 1$.

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By Theorem G2, $|Supp(A_t\ b_t)| > n_t$.

By this result and Theorem G1 $\xrightarrow{\text{implies}}$ $y_{is}^{-1} y_{jt} \cdots y_{kp}^{-1} y_{lq} = 1$

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Thank you!

[N, 2016] “Uncountable locally free groups and their group rings”
arXiv:1601.00295

[N and A, 2017] “Non-noetherian groups and primitivity of their group algebras”
J. Algebra Vol. 473

[N, 2011] “Group rings of countable non-abelian locally free groups are primitive”
Int. J. alg. and comp Vol 21

[N,2007] “Group rings of proper ascending HNN extensions of countably infinite free groups are primitive”
J. Algebra Vol. 317