Bisets, the double Burnside ring, and the subgroups of a direct product

Götz Pfeiffer

School of Mathematics, Statistics and Applied Mathematics
NUI, Galway

Groups St Andrews in Birmingham 2017

Bisets.

- Let G, H be finite groups, acting on a finite set X: $(g.x).h = g.(x.h)$, then ${}_GX_H$ is called a (G, H)-**biset**.

Example

$$ {}_GX_H = \{f\colon A \to B\}, $$

where $G \leq \mathrm{Sym}(A)$ and $H \leq \mathrm{Sym}(B)$, for finite sets A, B.

- Disjoint unions of (G, H)-bisets are (G, H)-bisets.
- The **double Burnside group** $B(G, H)$ is the Grothendieck group of the category of (G, H)-bisets, $[X] + [Y] = [X \amalg Y]$.
- Setting $g.x.h = x.(g^{-1}, h)$, yields $B(G, H) \cong B(G \times H)$, as abelian groups, with basis $\mathrm{Sub}(G \times H)/(G \times H)$.

Bisets.

- Let G, H be finite groups, acting on a finite set X: $(g.x).h = g.(x.h)$, then ${}_G X_H$ is called a (G, H)-**biset**.

Example

$${}_G X_H = \{f\colon A \to B\},$$

where $G \leq \mathrm{Sym}(A)$ and $H \leq \mathrm{Sym}(B)$, for finite sets A, B.

- Disjoint unions of (G, H)-bisets are (G, H)-bisets.
- The **double Burnside group** $B(G, H)$ is the Grothendieck group of the category of (G, H)-bisets, $[X] + [Y] = [X \amalg Y]$.
- Setting $g.x.h = x.(g^{-1}, h)$, yields $B(G, H) \cong B(G \times H)$, as abelian groups, with basis $\mathrm{Sub}(G \times H)/(G \times H)$.

Bisets.

- Let G, H be finite groups, acting on a finite set X: $(g.x).h = g.(x.h)$, then ${}_G X_H$ is called a (G, H)-**biset**.

Example

$${}_G X_H = \{f \colon A \to B\},$$

where $G \leq \mathrm{Sym}(A)$ and $H \leq \mathrm{Sym}(B)$, for finite sets A, B.

- Disjoint unions of (G, H)-bisets are (G, H)-bisets.
- The **double Burnside group** $B(G, H)$ is the Grothendieck group of the category of (G, H)-bisets, $[X] + [Y] = [X \amalg Y]$.
- Setting $g.x.h = x.(g^{-1}, h)$, yields $B(G, H) \cong B(G \times H)$, as abelian groups, with basis $\mathrm{Sub}(G \times H)/(G \times H)$.

Bisets.

- Let G, H be finite groups, acting on a finite set X: $(g.x).h = g.(x.h)$, then ${}_GX_H$ is called a (G, H)-**biset**.

Example

$${}_GX_H = \{f\colon A \to B\},$$

where $G \leq \mathrm{Sym}(A)$ and $H \leq \mathrm{Sym}(B)$, for finite sets A, B.

- Disjoint unions of (G, H)-bisets are (G, H)-bisets.
- The **double Burnside group** $B(G, H)$ is the Grothendieck group of the category of (G, H)-bisets, $[X] + [Y] = [X \amalg Y]$.
- Setting $g.x.h = x.(g^{-1}, h)$, yields $B(G, H) \cong B(G \times H)$, as abelian groups, with basis $\mathrm{Sub}(G \times H)/(G \times H)$.

Bisets.

- Let G, H be finite groups, acting on a finite set X: $(g.x).h = g.(x.h)$, then ${}_G X_H$ is called a (G, H)-**biset**.

Example

$$ {}_G X_H = \{f \colon A \to B\}, $$

where $G \leq \mathrm{Sym}(A)$ and $H \leq \mathrm{Sym}(B)$, for finite sets A, B.

- Disjoint unions of (G, H)-bisets are (G, H)-bisets.
- The **double Burnside group** $B(G, H)$ is the Grothendieck group of the category of (G, H)-bisets, $[X] + [Y] = [X \amalg Y]$.
- Setting $g.x.h = x.(g^{-1}, h)$, yields $B(G, H) \cong B(G \times H)$, as abelian groups, with basis $\mathrm{Sub}(G \times H)/(G \times H)$.

Tensor Products and the Double Burnside Ring.

- The **tensor product** of ${}_G X_H$ and ${}_H Y_K$ is the (G, K)-biset
$$X \times_H Y = (X \times Y)/H$$
of H-orbits under the action $(x, y).h = (x.h, h^{-1}.y)$.

Example

$$\{f : A \to B\} \times_{\mathrm{Sym}(B)} \{g : B \to C\} = \{g \circ f : A \to C\},$$
$$g \circ f = (g \circ \alpha) \circ (\alpha^{-1} \circ f),\ \alpha \in \mathrm{Sym}(B).$$

- The tensor product yields a bi-additive map $\_ \cdot_H \_ : B(G, H) \times B(H, K) \to B(G, K)$, mapping $([X], [Y])$ to
$$[X] \cdot_H [Y] = [X \times_H Y].$$
- In particular, $B(G, G)$ is a ring with multiplication $\cdot_G$.

Problem.

What is the structure of $\mathbb{Q}B(G, G) = \mathbb{Q} \otimes_{\mathbb{Z}} B(G, G)$, the **rational double Burnside algebra** of G?

Tensor Products and the Double Burnside Ring.

- The **tensor product** of ${}_G X_H$ and ${}_H Y_K$ is the (G, K)-biset
$$X \times_H Y = (X \times Y)/H$$
of H-orbits under the action $(x, y).h = (x.h, h^{-1}.y)$.

Example

$$\{f\colon A \to B\} \times_{\mathrm{Sym}(B)} \{g\colon B \to C\} = \{g \circ f\colon A \to C\},$$
$$g \circ f = (g \circ \alpha) \circ (\alpha^{-1} \circ f),\ \alpha \in \mathrm{Sym}(B).$$

- The tensor product yields a bi-additive map $\_ \cdot_H \_\colon B(G, H) \times B(H, K) \to B(G, K)$, mapping $([X], [Y])$ to
$$[X] \cdot_H [Y] = [X \times_H Y].$$
- In particular, $B(G, G)$ is a ring with multiplication $\cdot_G$.

Problem.

What is the structure of $\mathbb{Q}B(G, G) = \mathbb{Q} \otimes_{\mathbb{Z}} B(G, G)$, the **rational double Burnside algebra** of G?

Tensor Products and the Double Burnside Ring.

- The **tensor product** of ${}_GX_H$ and ${}_HY_K$ is the (G, K)-biset
$$X \times_H Y = (X \times Y)/H$$
of H-orbits under the action $(x, y).h = (x.h, h^{-1}.y)$.

Example

$$\{f: A \to B\} \times_{\mathrm{Sym}(B)} \{g: B \to C\} = \{g \circ f: A \to C\},$$
$$g \circ f = (g \circ \alpha) \circ (\alpha^{-1} \circ f),\ \alpha \in \mathrm{Sym}(B).$$

- The tensor product yields a bi-additive map $\_ \cdot_H \_: B(G, H) \times B(H, K) \to B(G, K)$, mapping $([X], [Y])$ to
$$[X] \cdot_H [Y] = [X \times_H Y].$$
- In particular, $B(G, G)$ is a ring with multiplication $\cdot_G$.

Problem.

What is the structure of $\mathbb{Q}B(G, G) = \mathbb{Q} \otimes_{\mathbb{Z}} B(G, G)$, the **rational double Burnside algebra** of G?

Tensor Products and the Double Burnside Ring.

- The **tensor product** of ${}_G X_H$ and ${}_H Y_K$ is the (G,K)-biset
$$X \times_H Y = (X \times Y)/H$$
of H-orbits under the action $(x,y).h = (x.h, h^{-1}.y)$.

Example

$$\{f\colon A \to B\} \times_{\mathrm{Sym}(B)} \{g\colon B \to C\} = \{g \circ f\colon A \to C\},$$
$$g \circ f = (g \circ \alpha) \circ (\alpha^{-1} \circ f),\ \alpha \in \mathrm{Sym}(B).$$

- The tensor product yields a bi-additive map $\_ \cdot_H \_\colon B(G,H) \times B(H,K) \to B(G,K)$, mapping $([X],[Y])$ to
$$[X] \cdot_H [Y] = [X \times_H Y].$$
- In particular, $B(G,G)$ is a ring with multiplication $\cdot_G$.

Problem.

What is the structure of $\mathbb{Q}B(G,G) = \mathbb{Q} \otimes_{\mathbb{Z}} B(G,G)$, the **rational double Burnside algebra** of G?

Tensor Products and the Double Burnside Ring.

- The **tensor product** of ${}_G X_H$ and ${}_H Y_K$ is the (G, K)-biset
$$X \times_H Y = (X \times Y)/H$$
of H-orbits under the action $(x, y).h = (x.h, h^{-1}.y)$.

Example

$$\{f\colon A \to B\} \times_{\mathrm{Sym}(B)} \{g\colon B \to C\} = \{g \circ f\colon A \to C\},$$
$$g \circ f = (g \circ \alpha) \circ (\alpha^{-1} \circ f),\ \alpha \in \mathrm{Sym}(B).$$

- The tensor product yields a bi-additive map $\_ \cdot_H \_\colon B(G, H) \times B(H, K) \to B(G, K)$, mapping $([X], [Y])$ to
$$[X] \cdot_H [Y] = [X \times_H Y].$$
- In particular, $B(G, G)$ is a ring with multiplication $\cdot_G$.

Problem.

What is the structure of $\mathbb{Q}B(G, G) = \mathbb{Q} \otimes_{\mathbb{Z}} B(G, G)$, the **rational double Burnside algebra** of G?

The Burnside Ring.

- As algebras, $\mathbb{Q}B(G, G) \not\cong \mathbb{Q}B(G \times G)$, since $\mathbb{Q}B(G, G)$ is **semisimple** iff G is cyclic (Boltje-Danz 2013).
- In contrast, $\mathbb{Q}B(G) = \mathbb{Q}^r = \bigoplus \mathbb{Q}^{1\times 1}$, $r = |\mathrm{Sub}(G)/G|$.
- Recall that the **Burnside ring** $B(G)$ is the Grothendieck group of the category of finite (right) G-sets, with multiplication $[X] \cdot [Y] = [X \times Y]$.
- The **mark homomorphism** $\beta_G \colon B(G) \to \mathbb{Z}^r$, $[X] \mapsto X^H$, $H \leq G$, embeds $B(G)$ into its **ghost ring** $\mathbb{Z}^r$.
- The **table of marks** $M(G)$ is the matrix of β_G relative to the **transitive** G-sets as basis of $B(G)$, and the standard basis of $\mathbb{Z}^r$.
- G is solvable $\iff$ $B(G)$ has only trivial idempotents ± 1 (Dress 1969).

The Burnside Ring.

- As algebras, $\mathbb{Q}B(G,G) \not\cong \mathbb{Q}B(G \times G)$, since $\mathbb{Q}B(G,G)$ is **semisimple** iff G is cyclic (Boltje-Danz 2013).
- In contrast, $\mathbb{Q}B(G) = \mathbb{Q}^r = \bigoplus \mathbb{Q}^{1\times 1}$, $r = |\mathrm{Sub}(G)/G|$.
- Recall that the **Burnside ring** $B(G)$ is the Grothendieck group of the category of finite (right) G-sets, with multiplication $[X] \cdot [Y] = [X \times Y]$.
- The **mark homomorphism** $\beta_G \colon B(G) \to \mathbb{Z}^r$, $[X] \mapsto X^H$, $H \leq G$, embeds $B(G)$ into its **ghost ring** $\mathbb{Z}^r$.
- The **table of marks** $M(G)$ is the matrix of β_G relative to the **transitive** G-sets as basis of $B(G)$, and the standard basis of $\mathbb{Z}^r$.
- G is solvable $\iff$ $B(G)$ has only trivial idempotents ± 1 (Dress 1969).

The Burnside Ring.

- As algebras, $\mathbb{Q}B(G, G) \not\cong \mathbb{Q}B(G \times G)$, since $\mathbb{Q}B(G, G)$ is **semisimple** iff G is cyclic (Boltje-Danz 2013).
- In contrast, $\mathbb{Q}B(G) = \mathbb{Q}^r = \bigoplus \mathbb{Q}^{1\times 1}$, $r = |\mathrm{Sub}(G)/G|$.
- Recall that the **Burnside ring** $B(G)$ is the Grothendieck group of the category of finite (right) G-sets, with multiplication $[X] \cdot [Y] = [X \times Y]$.
- The **mark homomorphism** $\beta_G \colon B(G) \to \mathbb{Z}^r$, $[X] \mapsto X^H$, $H \leq G$, embeds $B(G)$ into its **ghost ring** $\mathbb{Z}^r$.
- The **table of marks** $M(G)$ is the matrix of β_G relative to the **transitive** G-sets as basis of $B(G)$, and the standard basis of $\mathbb{Z}^r$.
- G is solvable $\iff$ $B(G)$ has only trivial idempotents ± 1 (Dress 1969).

The Burnside Ring.

- As algebras, $\mathbb{Q}B(G,G) \not\cong \mathbb{Q}B(G \times G)$, since $\mathbb{Q}B(G,G)$ is **semisimple** iff G is cyclic (Boltje-Danz 2013).
- In contrast, $\mathbb{Q}B(G) = \mathbb{Q}^r = \bigoplus \mathbb{Q}^{1\times 1}$, $r = |\mathrm{Sub}(G)/G|$.
- Recall that the **Burnside ring** $B(G)$ is the Grothendieck group of the category of finite (right) G-sets, with multiplication $[X] \cdot [Y] = [X \times Y]$.
- The **mark homomorphism** $\beta_G \colon B(G) \to \mathbb{Z}^r$, $[X] \mapsto X^H$, $H \leq G$, embeds $B(G)$ into its **ghost ring** $\mathbb{Z}^r$.
- The **table of marks** $M(G)$ is the matrix of β_G relative to the **transitive** G-sets as basis of $B(G)$, and the standard basis of $\mathbb{Z}^r$.
- G is solvable $\iff$ $B(G)$ has only trivial idempotents ± 1 (Dress 1969).

The Burnside Ring.

- As algebras, $\mathbb{Q}B(G,G) \not\cong \mathbb{Q}B(G \times G)$, since $\mathbb{Q}B(G,G)$ is **semisimple** iff G is cyclic (Boltje-Danz 2013).
- In contrast, $\mathbb{Q}B(G) = \mathbb{Q}^r = \bigoplus \mathbb{Q}^{1\times 1}$, $r = |\mathrm{Sub}(G)/G|$.
- Recall that the **Burnside ring** $B(G)$ is the Grothendieck group of the category of finite (right) G-sets, with multiplication $[X] \cdot [Y] = [X \times Y]$.
- The **mark homomorphism** $\beta_G \colon B(G) \to \mathbb{Z}^r$, $[X] \mapsto X^H$, $H \leq G$, embeds $B(G)$ into its **ghost ring** $\mathbb{Z}^r$.
- The **table of marks** $M(G)$ is the matrix of β_G relative to the **transitive** G-sets as basis of $B(G)$, and the standard basis of $\mathbb{Z}^r$.
- G is solvable $\iff$ $B(G)$ has only trivial idempotents ± 1 (Dress 1969).

The Burnside Ring.

- As algebras, $\mathbb{Q}B(G,G) \not\cong \mathbb{Q}B(G \times G)$, since $\mathbb{Q}B(G,G)$ is **semisimple** iff G is cyclic (Boltje-Danz 2013).
- In contrast, $\mathbb{Q}B(G) = \mathbb{Q}^r = \bigoplus \mathbb{Q}^{1\times 1}$, $r = |\mathrm{Sub}(G)/G|$.
- Recall that the **Burnside ring** $B(G)$ is the Grothendieck group of the category of finite (right) G-sets, with multiplication $[X] \cdot [Y] = [X \times Y]$.
- The **mark homomorphism** $\beta_G \colon B(G) \to \mathbb{Z}^r$, $[X] \mapsto X^H$, $H \leq G$, embeds $B(G)$ into its **ghost ring** $\mathbb{Z}^r$.
- The **table of marks** $M(G)$ is the matrix of β_G relative to the **transitive** G-sets as basis of $B(G)$, and the standard basis of $\mathbb{Z}^r$.
- G is solvable $\iff$ $B(G)$ has only trivial idempotents ± 1 (Dress 1969).

Example.

The table of marks of $G = \mathrm{Sym}_3$

$$M(G) = \left(\begin{array}{c|cccc} 1 & 6 & . & . & . \\ 2 & 3 & 1 & . & . \\ 3 & 2 & . & 2 & . \\ G & 1 & 1 & 1 & 1 \end{array}\right) = \begin{pmatrix} 6 & . & . & . \\ . & 1 & . & . \\ . & . & 2 & . \\ . & . & . & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & . & . & . \\ 3 & 1 & . & . \\ 1 & . & 1 & . \\ 1 & 1 & 1 & 1 \end{pmatrix}$$
$$= \mathbf{D}_0 \cdot \mathbf{A}(\leq)$$

- $|(G/H)^{H'}| = |N_G(H) : H| \cdot \#\{H' \leq H^g : g \in G\}$.
- $\mathbf{A}(\leq)$ is the **class incidence matrix** of poset $(\mathrm{Sub}(G), \leq)$.

$\rightsquigarrow$ Plan.

Find a base change matrix $M'(G \times G)$, relative to a suitable partial order, that reveals the structure of $\mathbb{Q}B(G, G)$.

Example.

The table of marks of $G = \mathrm{Sym}_3$

$$M(G) = \left(\begin{array}{c|cccc} 1 & 6 & . & . & . \\ 2 & 3 & 1 & . & . \\ 3 & 2 & . & 2 & . \\ G & 1 & 1 & 1 & 1 \end{array}\right) = \begin{pmatrix} 6 & . & . & . \\ . & 1 & . & . \\ . & . & 2 & . \\ . & . & . & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & . & . & . \\ 3 & 1 & . & . \\ 1 & . & 1 & . \\ 1 & 1 & 1 & 1 \end{pmatrix}$$
$$= \mathbf{D}_0 \cdot \mathbf{A}(\leq)$$

- $|(G/H)^{H'}| = |N_G(H) : H| \cdot \#\{H' \leq H^g : g \in G\}$.
- $\mathbf{A}(\leq)$ is the **class incidence matrix** of poset $(\mathrm{Sub}(G), \leq)$.

$\rightsquigarrow$ Plan.

Find a base change matrix $M'(G \times G)$, relative to a suitable partial order, that reveals the structure of $\mathbb{Q}B(G, G)$.

Example.

The table of marks of $G = \mathrm{Sym}_3$

$$M(G) = \begin{array}{c|c} 1 & \\ 2 & \\ 3 & \\ G & \end{array}\!\!\begin{pmatrix} 6 & . & . & . \\ 3 & 1 & . & . \\ 2 & . & 2 & . \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & . & . & . \\ . & 1 & . & . \\ . & . & 2 & . \\ . & . & . & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & . & . & . \\ 3 & 1 & . & . \\ 1 & . & 1 & . \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$= \mathbf{D}_0 \cdot \mathbf{A}(\leq)$$

- $|(G/H)^{H'}| = |N_G(H) : H| \cdot \#\{H' \leq H^g : g \in G\}$.
- $\mathbf{A}(\leq)$ is the **class incidence matrix** of poset $(\mathrm{Sub}(G), \leq)$.

$\rightsquigarrow$ Plan.

Find a base change matrix $M'(G \times G)$, relative to a suitable partial order, that reveals the structure of $\mathbb{Q}B(G, G)$.

Example.

The table of marks of $G = \mathrm{Sym}_3$

$$M(G) = \left(\begin{array}{c|cccc} 1 & 6 & . & . & . \\ 2 & 3 & 1 & . & . \\ 3 & 2 & . & 2 & . \\ G & 1 & 1 & 1 & 1 \end{array}\right) = \begin{pmatrix} 6 & . & . & . \\ . & 1 & . & . \\ . & . & 2 & . \\ . & . & . & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & . & . & . \\ 3 & 1 & . & . \\ 1 & . & 1 & . \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$= \mathbf{D}_0 \cdot \mathbf{A}(\leq)$$

- $|(G/H)^{H'}| = |N_G(H) : H| \cdot \#\{H' \leq H^g : g \in G\}$.
- $\mathbf{A}(\leq)$ is the **class incidence matrix** of poset $(\mathrm{Sub}(G), \leq)$.

$\rightsquigarrow$ Plan.

Find a base change matrix $M'(G \times G)$, relative to a suitable partial order, that reveals the structure of $\mathbb{Q}B(G, G)$.

Subgroups of a Direct Product.

Theorem (Goursat's Lemma, 1889)

Subgroups of $G_1 \times G_2$ are isomorphisms between a section of G_1 and a section of G_2.

- A subgroup $L \leq G_1 \times G_2$ is a **difunctional relation**, the **graph** of a bijection between section quotients P_i/K_i of G_i:
$$L = \{(p_1, p_2) \in P_1 \times P_2 : (K_1 p_1)^\theta = K_2 p_2\}.$$
- Write $p_i(L) = P_i$, $k_i(L) = K_i$ and $L = (\theta\colon P_1/K_1 \to P_2/K_2)$.
- If $L \leq G_1 \times G_2$ and $M \leq G_2 \times G_3$ then the relation product
$$L \circ M = \{(g,k) : (g,h) \in L,\ (h,k) \in M \text{ for some } h \in G_2\}$$
is a subgroup of $G_1 \times G_3$.

Theorem (Bouc)

$$[(G_1 \times G_2)/L] \cdot_{G_2} [(G_2 \times G_3)/M] = \sum_{d \in D} [(G_1 \times G_2)/(L^{(1,d)} \circ M)].$$

Subgroups of a Direct Product.

Theorem (Goursat's Lemma, 1889)

Subgroups of $G_1 \times G_2$ are isomorphisms between a section of G_1 and a section of G_2.

- A subgroup $L \leq G_1 \times G_2$ is a **difunctional relation**, the **graph** of a bijection between section quotients P_i/K_i of G_i:
$$L = \{(p_1, p_2) \in P_1 \times P_2 : (K_1 p_1)^\theta = K_2 p_2\}.$$
- Write $p_i(L) = P_i$, $k_i(L) = K_i$ and $L = (\theta\colon P_1/K_1 \to P_2/K_2)$.
- If $L \leq G_1 \times G_2$ and $M \leq G_2 \times G_3$ then the relation product
$$L \circ M = \{(g, k) : (g, h) \in L,\ (h, k) \in M \text{ for some } h \in G_2\}$$
is a subgroup of $G_1 \times G_3$.

Theorem (Bouc)

$$[(G_1 \times G_2)/L] \cdot_{G_2} [(G_2 \times G_3)/M] = \sum_{d \in D} [(G_1 \times G_2)/(L^{(1,d)} \circ M)].$$

Subgroups of a Direct Product.

Theorem (Goursat's Lemma, 1889)

Subgroups of $G_1 \times G_2$ are isomorphisms between a section of G_1 and a section of G_2.

- A subgroup $L \leq G_1 \times G_2$ is a **difunctional relation**, the **graph** of a bijection between section quotients P_i/K_i of G_i:
$$L = \{(p_1, p_2) \in P_1 \times P_2 : (K_1 p_1)^\theta = K_2 p_2\}.$$
- Write $p_i(L) = P_i$, $k_i(L) = K_i$ and $L = (\theta\colon P_1/K_1 \to P_2/K_2)$.
- If $L \leq G_1 \times G_2$ and $M \leq G_2 \times G_3$ then the relation product
$$L \circ M = \{(g, k) : (g, h) \in L,\ (h, k) \in M \text{ for some } h \in G_2\}$$
is a subgroup of $G_1 \times G_3$.

Theorem (Bouc)

$$[(G_1 \times G_2)/L] \cdot_{G_2} [(G_2 \times G_3)/M] = \sum_{d \in D} [(G_1 \times G_2)/(L^{(1,d)} \circ M)].$$

Subgroups of a Direct Product.

Theorem (Goursat's Lemma, 1889)

Subgroups of $G_1 \times G_2$ are isomorphisms between a section of G_1 and a section of G_2.

- A subgroup $L \leq G_1 \times G_2$ is a **difunctional relation**, the **graph** of a bijection between section quotients P_i/K_i of G_i:
$$L = \{(p_1, p_2) \in P_1 \times P_2 : (K_1 p_1)^\theta = K_2 p_2\}.$$
- Write $p_i(L) = P_i$, $k_i(L) = K_i$ and $L = (\theta\colon P_1/K_1 \to P_2/K_2)$.
- If $L \leq G_1 \times G_2$ and $M \leq G_2 \times G_3$ then the relation product
$$L \circ M = \{(g, k) : (g, h) \in L,\ (h, k) \in M \text{ for some } h \in G_2\}$$
is a subgroup of $G_1 \times G_3$.

Theorem (Bouc)

$[(G_1 \times G_2)/L] \cdot_{G_2} [(G_2 \times G_3)/M] = \sum_{d \in D} [(G_1 \times G_2)/(L^{(1,d)} \circ M)]$.

Subgroups of a Direct Product.

Theorem (Goursat's Lemma, 1889)

Subgroups of $G_1 \times G_2$ are isomorphisms between a section of G_1 and a section of G_2.

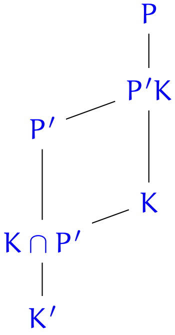
- A subgroup $L \leq G_1 \times G_2$ is a **difunctional relation**, the **graph** of a bijection between section quotients P_i/K_i of G_i:
$$L = \{(p_1, p_2) \in P_1 \times P_2 : (K_1 p_1)^\theta = K_2 p_2\}.$$
- Write $p_i(L) = P_i$, $k_i(L) = K_i$ and $L = (\theta\colon P_1/K_1 \to P_2/K_2)$.
- If $L \leq G_1 \times G_2$ and $M \leq G_2 \times G_3$ then the relation product
$$L \circ M = \{(g, k) : (g, h) \in L,\ (h, k) \in M \text{ for some } h \in G_2\}$$
is a subgroup of $G_1 \times G_3$.

Theorem (Bouc)

$[(G_1 \times G_2)/L] \cdot_{G_2} [(G_2 \times G_3)/M] = \sum_{d \in D} [(G_1 \times G_2)/(L^{(1,d)} \circ M)]$.

The Sections Lattice.

- $\mathrm{Sec}(G) := \{(P, K) : K \trianglelefteq P \leq G\}$, the set of **all sections** of G.
- Set $(P', K') \leq (P, K) :\Longleftrightarrow P' \leq P$ and $K' \leq K$.
- $(\mathrm{Sec}(G), \leq)$ inherits the **lattice** property from $(\mathrm{Sub}(G), \leq)$.

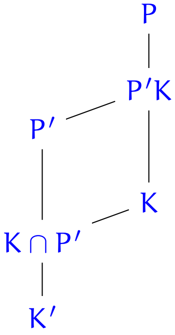


Central Observation

- If $(P', K') \leq (P, K)$ then $K'p \mapsto Kp$ is a homomorphism $\phi \colon P'/K' \to P/K$.
- Like every homomorphism, $\phi = \phi_3 \circ \phi_2 \circ \phi_1$, where $\phi_1 \colon P'/K' \to P'/(K \cap P')$ is **epi**, $\phi_2 \colon P'/(K \cap P') \to P'K/K$ is **iso**, and $\phi_3 \colon P'K/K \to P/K$ is **mono**.
- Hence, there exist two **uniquely determined** intermediate sections!

The Sections Lattice.

- $\mathrm{Sec}(G) := \{(P, K) : K \trianglelefteq P \leq G\}$, the set of **all sections** of G.
- Set $(P', K') \leq (P, K) :\Longleftrightarrow P' \leq P$ and $K' \leq K$.
- $(\mathrm{Sec}(G), \leq)$ inherits the **lattice** property from $(\mathrm{Sub}(G), \leq)$.

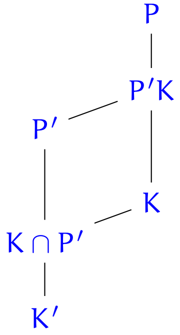


Central Observation

- If $(P', K') \leq (P, K)$ then $K'p \mapsto Kp$ is a homomorphism $\phi \colon P'/K' \to P/K$.
- Like every homomorphism, $\phi = \phi_3 \circ \phi_2 \circ \phi_1$, where $\phi_1 \colon P'/K' \to P'/(K \cap P')$ is **epi**, $\phi_2 \colon P'/(K \cap P') \to P'K/K$ is **iso**, and $\phi_3 \colon P'K/K \to P/K$ is **mono**.
- Hence, there exist two **uniquely determined** intermediate sections!

The Sections Lattice.

- $\mathrm{Sec}(G) := \{(P, K) : K \trianglelefteq P \leq G\}$, the set of **all sections** of G.
- Set $(P', K') \leq (P, K) :\Longleftrightarrow P' \leq P$ and $K' \leq K$.
- $(\mathrm{Sec}(G), \leq)$ inherits the **lattice** property from $(\mathrm{Sub}(G), \leq)$.

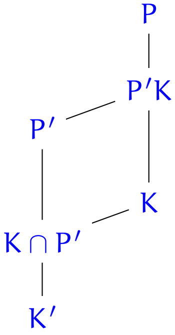


Central Observation

- If $(P', K') \leq (P, K)$ then $K'p \mapsto Kp$ is a homomorphism $\phi: P'/K' \to P/K$.
- Like every homomorphism, $\phi = \phi_3 \circ \phi_2 \circ \phi_1$, where $\phi_1: P'/K' \to P'/(K \cap P')$ is **epi**, $\phi_2: P'/(K \cap P') \to P'K/K$ is **iso**, and $\phi_3: P'K/K \to P/K$ is **mono**.
- Hence, there exist two **uniquely determined** intermediate sections!

The Sections Lattice.

- $\mathrm{Sec}(G) := \{(P, K) : K \trianglelefteq P \leq G\}$, the set of **all sections** of G.
- Set $(P', K') \leq (P, K) :\Longleftrightarrow P' \leq P$ and $K' \leq K$.
- $(\mathrm{Sec}(G), \leq)$ inherits the **lattice** property from $(\mathrm{Sub}(G), \leq)$.



Central Observation

- If $(P', K') \leq (P, K)$ then $K'p \mapsto Kp$ is a homomorphism $\phi\colon P'/K' \to P/K$.
- Like every homomorphism, $\phi = \phi_3 \circ \phi_2 \circ \phi_1$, where $\phi_1 \colon P'/K' \to P'/(K \cap P')$ is **epi**, $\phi_2 \colon P'/(K \cap P') \to P'K/K$ is **iso**, and $\phi_3 \colon P'K/K \to P/K$ is **mono**.
- Hence, there exist two **uniquely determined** intermediate sections!

The Sections Lattice.

- $\mathrm{Sec}(G) := \{(P, K) : K \trianglelefteq P \le G\}$, the set of **all sections** of G.
- Set $(P', K') \le (P, K) :\Longleftrightarrow P' \le P$ and $K' \le K$.
- $(\mathrm{Sec}(G), \le)$ inherits the **lattice** property from $(\mathrm{Sub}(G), \le)$.

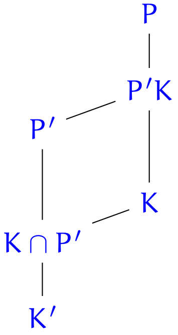
$$\begin{tikzcd}
 & P \arrow[d, dash] \\
 & P'K \arrow[dl, dash] \arrow[dd, dash] \\
P' \arrow[dd, dash] & \\
 & K \arrow[dl, dash] \\
K \cap P' \arrow[d, dash] & \\
K' &
\end{tikzcd}$$

Central Observation

- If $(P', K') \le (P, K)$ then $K'p \mapsto Kp$ is a homomorphism $\phi\colon P'/K' \to P/K$.
- Like every homomorphism, $\phi = \phi_3 \circ \phi_2 \circ \phi_1$, where $\phi_1\colon P'/K' \to P'/(K \cap P')$ is **epi**, $\phi_2\colon P'/(K \cap P') \to P'K/K$ is **iso**, and $\phi_3\colon P'K/K \to P/K$ is **mono**.
- Hence, there exist two **uniquely determined** intermediate sections!

The Sections Lattice.

- $\mathrm{Sec}(G) := \{(P, K) : K \trianglelefteq P \leq G\}$, the set of **all sections** of G.
- Set $(P', K') \leq (P, K) :\Longleftrightarrow P' \leq P$ and $K' \leq K$.
- $(\mathrm{Sec}(G), \leq)$ inherits the **lattice** property from $(\mathrm{Sub}(G), \leq)$.

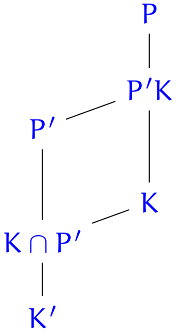


Central Observation

- If $(P', K') \leq (P, K)$ then $K'p \mapsto Kp$ is a homomorphism $\phi\colon P'/K' \to P/K$.
- Like every homomorphism, $\phi = \phi_3 \circ \phi_2 \circ \phi_1$, where
  $\phi_1\colon P'/K' \to P'/(K \cap P')$ is **epi**,
  $\phi_2\colon P'/(K \cap P') \to P'K/K$ is **iso**, and
  $\phi_3\colon P'K/K \to P/K$ is **mono**.
- Hence, there exist two **uniquely determined** intermediate sections!

The Sections Lattice.

- $\mathrm{Sec}(G) := \{(P, K) : K \trianglelefteq P \leq G\}$, the set of **all sections** of G.
- Set $(P', K') \leq (P, K) :\Longleftrightarrow P' \leq P$ and $K' \leq K$.
- $(\mathrm{Sec}(G), \leq)$ inherits the **lattice** property from $(\mathrm{Sub}(G), \leq)$.



Central Observation

- If $(P', K') \leq (P, K)$ then $K'p \mapsto Kp$ is a homomorphism $\phi \colon P'/K' \to P/K$.
- Like every homomorphism, $\phi = \phi_3 \circ \phi_2 \circ \phi_1$, where $\phi_1 \colon P'/K' \to P'/(K \cap P')$ is **epi**, $\phi_2 \colon P'/(K \cap P') \to P'K/K$ is **iso**, and $\phi_3 \colon P'K/K \to P/K$ is **mono**.
- Hence, there exist two **uniquely determined** intermediate sections!

A Decomposition of the Class Incidence Matrix.

- Accordingly, $\leq \; = \; \leq_K \circ \leq_{P/K} \circ \leq_P$, where:

$$(P', K') \leq_K (P, K) :\Longleftrightarrow \; K = K' \text{ and } P' \leq P;$$
$$(P', K') \leq_P (P, K) :\Longleftrightarrow \; P = P' \text{ and } K' \leq K;$$
$$(P', K') \leq_{P/K} (P, K) :\Longleftrightarrow \; (P', K') \leq (P, K),\ P'/K' \cong P/K.$$

Theorem (Masterson-Pf, 2017)

$$\mathbf{A}(\leq) = \mathbf{A}(\leq_K) \cdot \mathbf{A}(\leq_{P/K}) \cdot \mathbf{A}(\leq_P),$$

where

- $\mathbf{A}(\leq_K) = \bigoplus_{[K]} \mathbf{A}_K(\leq)$, $(N_G(K) \curvearrowright \{P \in \mathrm{Sub}(G) : K \trianglelefteq P\})$;
- $\mathbf{A}(\leq_P) = \bigoplus_{[P]} \mathbf{A}_P(\leq)$, $(N_G(P) \curvearrowright \{K \in \mathrm{Sub}(G) : K \trianglelefteq P\})$;
- $\mathbf{A}(\leq_{P/K}) = \bigoplus_{U \sqsubseteq G} \mathbf{A}_U(\leq_{P/K})$.

A Decomposition of the Class Incidence Matrix.

- Accordingly, $\leq \; = \; \leq_K \circ \leq_{P/K} \circ \leq_P$, where:

$$(P', K') \leq_K (P, K) :\iff K = K' \text{ and } P' \leq P;$$
$$(P', K') \leq_P (P, K) :\iff P = P' \text{ and } K' \leq K;$$
$$(P', K') \leq_{P/K} (P, K) :\iff (P', K') \leq (P, K),\ P'/K' \cong P/K.$$

Theorem (Masterson-Pf, 2017)

$$\mathbf{A}(\leq) = \mathbf{A}(\leq_K) \cdot \mathbf{A}(\leq_{P/K}) \cdot \mathbf{A}(\leq_P),$$

where

- $\mathbf{A}(\leq_K) = \bigoplus_{[K]} \mathbf{A}_K(\leq)$, $(N_G(K) \curvearrowright \{P \in \mathrm{Sub}(G) : K \trianglelefteq P\})$;
- $\mathbf{A}(\leq_P) = \bigoplus_{[P]} \mathbf{A}_P(\leq)$, $(N_G(P) \curvearrowright \{K \in \mathrm{Sub}(G) : K \trianglelefteq P\})$;
- $\mathbf{A}(\leq_{P/K}) = \bigoplus_{U \sqsubseteq G} \mathbf{A}_U(\leq_{P/K})$.

A Decomposition of the Class Incidence Matrix.

- Accordingly, $\leq \, = \, \leq_K \circ \leq_{P/K} \circ \leq_P$, where:

$$(P', K') \leq_K (P, K) :\iff K = K' \text{ and } P' \leq P;$$
$$(P', K') \leq_P (P, K) :\iff P = P' \text{ and } K' \leq K;$$
$$(P', K') \leq_{P/K} (P, K) :\iff (P', K') \leq (P, K),\ P'/K' \cong P/K.$$

Theorem (Masterson-Pf, 2017)

$$\mathbf{A}(\leq) = \mathbf{A}(\leq_K) \cdot \mathbf{A}(\leq_{P/K}) \cdot \mathbf{A}(\leq_P),$$

where

- $\mathbf{A}(\leq_K) = \bigoplus_{[K]} \mathbf{A}_K(\leq)$, $(N_G(K) \curvearrowright \{P \in \mathrm{Sub}(G) : K \trianglelefteq P\})$;
- $\mathbf{A}(\leq_P) = \bigoplus_{[P]} \mathbf{A}_P(\leq)$, $(N_G(P) \curvearrowright \{K \in \mathrm{Sub}(G) : K \trianglelefteq P\})$;
- $\mathbf{A}(\leq_{P/K}) = \bigoplus_{U \sqsubseteq G} \mathbf{A}_U(\leq_{P/K})$.

A Decomposition of the Class Incidence Matrix.

- Accordingly, $\leq \; = \; \leq_K \circ \leq_{P/K} \circ \leq_P$, where:

$$(P', K') \leq_K (P, K) :\iff K = K' \text{ and } P' \leq P;$$
$$(P', K') \leq_P (P, K) :\iff P = P' \text{ and } K' \leq K;$$
$$(P', K') \leq_{P/K} (P, K) :\iff (P', K') \leq (P, K),\ P'/K' \cong P/K.$$

Theorem (Masterson-Pf, 2017)

$$\mathbf{A}(\leq) = \mathbf{A}(\leq_K) \cdot \mathbf{A}(\leq_{P/K}) \cdot \mathbf{A}(\leq_P),$$

where

- $\mathbf{A}(\leq_K) = \bigoplus_{[K]} \mathbf{A}_K(\leq)$, $(N_G(K) \curvearrowright \{P \in \mathrm{Sub}(G) : K \trianglelefteq P\})$;
- $\mathbf{A}(\leq_P) = \bigoplus_{[P]} \mathbf{A}_P(\leq)$, $(N_G(P) \curvearrowright \{K \in \mathrm{Sub}(G) : K \trianglelefteq P\})$;
- $\mathbf{A}(\leq_{P/K}) = \bigoplus_{U \sqsubseteq G} \mathbf{A}_U(\leq_{P/K})$.

A Decomposition of the Class Incidence Matrix.

- Accordingly, $\leq \,=\, \leq_K \circ \leq_{P/K} \circ \leq_P$, where:

$$(P', K') \leq_K (P, K) :\Longleftrightarrow\ K = K' \text{ and } P' \leq P;$$
$$(P', K') \leq_P (P, K) :\Longleftrightarrow\ P = P' \text{ and } K' \leq K;$$
$$(P', K') \leq_{P/K} (P, K) :\Longleftrightarrow\ (P', K') \leq (P, K),\ P'/K' \cong P/K.$$

Theorem (Masterson-Pf, 2017)

$$\mathbf{A}(\leq) = \mathbf{A}(\leq_K) \cdot \mathbf{A}(\leq_{P/K}) \cdot \mathbf{A}(\leq_P),$$

where

- $\mathbf{A}(\leq_K) = \bigoplus_{[K]} \mathbf{A}_K(\leq)$, $(N_G(K) \curvearrowright \{P \in \mathrm{Sub}(G) : K \trianglelefteq P\})$;
- $\mathbf{A}(\leq_P) = \bigoplus_{[P]} \mathbf{A}_P(\leq)$, $(N_G(P) \curvearrowright \{K \in \mathrm{Sub}(G) : K \trianglelefteq P\})$;
- $\mathbf{A}(\leq_{P/K}) = \bigoplus_{U \sqsubseteq G} \mathbf{A}_U(\leq_{P/K})$.

Example $G = \mathrm{Sym}_3$.

$$
\mathbf{A}(\leq_P) = \begin{array}{c|c|cc|cc|ccc}
(1,1) & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
(2,1) & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
(2,2) & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
(3,1) & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
(3,3) & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\
\hline
(G,1) & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
(G,3) & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\
(G,G) & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\
\hline
 & (1,1) & (2,1) & (2,2) & (3,1) & (3,3) & (G,1) & (G,3) & (G,G)
\end{array}
$$

$$
\mathbf{A}(\leq_K) = \begin{array}{c|cccc|c|cc|c}
(1,1) & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
(2,1) & 3 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
(3,1) & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
(G,1) & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\hline
(2,2) & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\hline
(3,3) & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
(G,3) & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\
\hline
(G,G) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\hline
 & (1,1) & (2,1) & (3,1) & (G,1) & (2,2) & (3,3) & (G,3) & (G,G)
\end{array}
$$

Example $G = \mathrm{Sym}_3$, cont'd.

$$\mathbf{A}(\leq_{P/K}) = \begin{array}{c|cccc|cc|c|c}
(1,1) & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
(2,2) & 3 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
(3,3) & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
(G,G) & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\hline
(2,1) & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
(G,3) & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
\hline
(3,1) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\hline
(G,1) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\hline
 & (1,1) & (2,2) & (3,3) & (G,G) & (2,1) & (G,3) & (3,1) & (G,1)
\end{array}$$

$$\mathbf{A}(\leq) = \begin{array}{c|cccc|cc|c|c}
(1,1) & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
(2,2) & 3 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
(3,3) & 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\
(G,G) & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
(2,1) & 3 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
(G,3) & 1 & \cdot & 1 & \cdot & 1 & 1 & 1 & 1 \\
\hline
(3,1) & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\hline
(G,1) & 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 \\
\hline
 & (1,1) & (2,2) & (3,3) & (G,G) & (2,1) & (G,3) & (3,1) & (G,1)
\end{array}$$

Conjugacy Classes of Subgroups.

- For $U \sqsubseteq G$, set $\mathrm{Mor}_U(G) := {}_G\{\theta\colon P/K \xrightarrow{\sim} U\}_{\mathrm{Aut}(U)}$ (biset!).
- Then $\mathrm{Sub}(G_1 \times G_2)/(G_1 \times G_2) =$
$$\coprod_U \mathrm{Mor}_U(G_1)/G_1 \times_{\mathrm{Aut}(U)} (\mathrm{Mor}_U(G_2)/G_2)^{\mathrm{op}}$$

Example $G = \mathrm{Sym}_3$.

- In $G \times G$, the effects of $\mathrm{Aut}(U)$ are trivial.
- $G \times G$ has 22 classes of subgroups.

$1/1$	1	2	3	4				
$2/2$	5	6	7	8				
$3/3$	9	10	11	12				
G/G	13	14	15	16				
$2/1$					17	18		
$G/3$					19	20		
$3/1$							21	
$G/1$								22

Conjugacy Classes of Subgroups.

- For $U \sqsubseteq G$, set $\mathrm{Mor}_U(G) := {}_G\{\theta\colon P/K \xrightarrow{\sim} U\}_{\mathrm{Aut}(U)}$ (biset!).
- Then $\mathrm{Sub}(G_1 \times G_2)/(G_1 \times G_2) =$

$$\coprod_U \mathrm{Mor}_U(G_1)/G_1 \times_{\mathrm{Aut}(U)} (\mathrm{Mor}_U(G_2)/G_2)^{\mathrm{op}}$$

Example $G = \mathrm{Sym}_3$.

- In $G \times G$, the effects of $\mathrm{Aut}(U)$ are trivial.
- $G \times G$ has 22 classes of subgroups.

$1/1$	1	2	3	4				
$2/2$	5	6	7	8				
$3/3$	9	10	11	12				
G/G	13	14	15	16				
$2/1$					17	18		
$G/3$					19	20		
$3/1$							21	
$G/1$								22

Conjugacy Classes of Subgroups.

- For $U \sqsubseteq G$, set $\mathrm{Mor}_U(G) := {}_G\{\theta\colon P/K \xrightarrow{\sim} U\}_{\mathrm{Aut}(U)}$ (biset!).
- Then $\mathrm{Sub}(G_1 \times G_2)/(G_1 \times G_2) =$
$$\coprod_U \mathrm{Mor}_U(G_1)/G_1 \times_{\mathrm{Aut}(U)} (\mathrm{Mor}_U(G_2)/G_2)^{\mathrm{op}}$$

Example $G = \mathrm{Sym}_3$.

- In $G \times G$, the effects of $\mathrm{Aut}(U)$ are trivial.
- $G \times G$ has 22 classes of subgroups.

$1/1$	1	2	3	4				
$2/2$	5	6	7	8				
$3/3$	9	10	11	12				
G/G	13	14	15	16				
$2/1$					17	18		
$G/3$					19	20		
$3/1$							21	
$G/1$								22

Conjugacy Classes of Subgroups.

- For $U \sqsubseteq G$, set $\mathrm{Mor}_U(G) := {}_G\{\theta\colon P/K \xrightarrow{\sim} U\}_{\mathrm{Aut}(U)}$ (biset!).
- Then $\mathrm{Sub}(G_1 \times G_2)/(G_1 \times G_2) =$

$$\coprod_U \mathrm{Mor}_U(G_1)/G_1 \times_{\mathrm{Aut}(U)} (\mathrm{Mor}_U(G_2)/G_2)^{\mathrm{op}}$$

Example $G = \mathrm{Sym}_3$.

- In $G \times G$, the effects of $\mathrm{Aut}(U)$ are trivial.
- $G \times G$ has 22 classes of subgroups.

$1/1$	1	2	3	4				
$2/2$	5	6	7	8				
$3/3$	9	10	11	12				
G/G	13	14	15	16				
$2/1$					17	18		
$G/3$					19	20		
$3/1$							21	
$G/1$								22

Conjugacy Classes of Subgroups.

- For $U \sqsubseteq G$, set $\mathrm{Mor}_U(G) := {}_G\{\theta\colon P/K \xrightarrow{\sim} U\}_{\mathrm{Aut}(U)}$ (biset!).
- Then $\mathrm{Sub}(G_1 \times G_2)/(G_1 \times G_2) =$
$$\coprod_U \mathrm{Mor}_U(G_1)/G_1 \times_{\mathrm{Aut}(U)} (\mathrm{Mor}_U(G_2)/G_2)^{\mathrm{op}}$$

Example $G = \mathrm{Sym}_3$.

- In $G \times G$, the effects of $\mathrm{Aut}(U)$ are trivial.
- $G \times G$ has 22 classes of subgroups.

$1/1$	1	2	3	4				
$2/2$	5	6	7	8				
$3/3$	9	10	11	12				
G/G	13	14	15	16				
$2/1$					17	18		
$G/3$					19	20		
$3/1$							21	
$G/1$								22

Subgroup Lattice.

Theorem

$$(\theta'\colon P_1'/K_1' \to P_2'/K_2') \leq (\theta\colon P_1/K_1 \to P_2/K_2)$$

iff $(P_i', K_i') \leq (P_i, K_i)$ *and the diagram commutes:*

$$\begin{tikzcd} P_1/K_1 \arrow[r, "\theta"] & P_2/K_2 \\ P_1'/K_1' \arrow[u, "\phi_1"] \arrow[r, "\theta'"] & P_2'/K_2' \arrow[u, "\phi_2"'] \end{tikzcd}$$

- As in the case of sections, for $X = K, P$ and P/K, define a partial order $\leq_X$ on $\mathrm{Sub}(G_1 \times G_2)$ as:

$$\theta' \leq_X \theta :\Longleftrightarrow \ \theta' \leq \theta \text{ and } (P_i', K_i') \leq_X (P_i, K_i),\ i = 1, 2.$$

Subgroup Lattice.

Theorem

$$(\theta'\colon P_1'/K_1' \to P_2'/K_2') \le (\theta\colon P_1/K_1 \to P_2/K_2)$$

iff $(P_i', K_i') \le (P_i, K_i)$ *and the diagram commutes:*

$$\begin{tikzcd} P_1/K_1 \arrow[r, "\theta"] & P_2/K_2 \\ P_1'/K_1' \arrow[u, "\phi_1"] \arrow[r, "\theta'"] & P_2'/K_2' \arrow[u, "\phi_2"'] \end{tikzcd}$$

- As in the case of sections, for $X = K, P$ and P/K, define a partial order $\le_X$ on $\mathrm{Sub}(G_1 \times G_2)$ as:

$$\theta' \le_X \theta :\Longleftrightarrow \theta' \le \theta \text{ and } (P_i', K_i') \le_X (P_i, K_i),\ i = 1, 2.$$

Table of Marks.

- Again, $\leq \, = \, \leq_K \circ \leq_{P/K} \circ \leq_P$, on $\mathrm{Sub}(G_1 \times G_2)$.

Theorem (Masterson-Pf. 2017)

$$M(G_1 \times G_2) = \mathbf{D}_0 \cdot \mathbf{A}(\leq_K) \cdot \mathbf{A}(\leq_{P/K}) \cdot \mathbf{A}(\leq_P),$$

where:

- $\mathbf{A}(\leq_K) = \bigoplus_{[K_1]} \bigoplus_{[K_2]} \mathbf{A}_{K_1,K_2}(\leq)$,
  $N_{G_1}(K_1) \times N_{G_2}(K_2) \curvearrowright \{L \in \mathrm{Sub}(G_1 \times G_2) : k_i(L) = K_i\}$;
- $\mathbf{A}(\leq_P) = \bigoplus_{[P_1]} \bigoplus_{[P_2]} \mathbf{A}_{P_1,P_2}(\leq)$,
  $N_{G_1}(P_1) \times N_{G_2}(P_2) \curvearrowright \{L \in \mathrm{Sub}(G_1 \times G_2) : p_i(L) = P_i\}$;
- $\mathbf{A}(\leq_{P/K}) = \bigoplus_{U} \mathbf{A}_U^{G_1}(\leq) \otimes_{\mathrm{Aut}U} \mathbf{A}_U^{G_2}(\leq)$, $G_i \curvearrowright \mathrm{Mor}_U(G_i)$.

Table of Marks.

- Again, $\leq = \leq_K \circ \leq_{P/K} \circ \leq_P$, on $\mathrm{Sub}(G_1 \times G_2)$.

Theorem (Masterson-Pf. 2017)

$$M(G_1 \times G_2) = \mathbf{D}_0 \cdot \mathbf{A}(\leq_K) \cdot \mathbf{A}(\leq_{P/K}) \cdot \mathbf{A}(\leq_P),$$

where:

- $\mathbf{A}(\leq_K) = \bigoplus_{[K_1]} \bigoplus_{[K_2]} \mathbf{A}_{K_1,K_2}(\leq)$,
  $N_{G_1}(K_1) \times N_{G_2}(K_2) \curvearrowright \{L \in \mathrm{Sub}(G_1 \times G_2) : k_i(L) = K_i\}$;
- $\mathbf{A}(\leq_P) = \bigoplus_{[P_1]} \bigoplus_{[P_2]} \mathbf{A}_{P_1,P_2}(\leq)$,
  $N_{G_1}(P_1) \times N_{G_2}(P_2) \curvearrowright \{L \in \mathrm{Sub}(G_1 \times G_2) : p_i(L) = P_i\}$;
- $\mathbf{A}(\leq_{P/K}) = \bigoplus_{U} \mathbf{A}_U^{G_1}(\leq) \otimes_{\mathrm{Aut}U} \mathbf{A}_U^{G_2}(\leq)$, $G_i \curvearrowright \mathrm{Mor}_U(G_i)$.

Table of Marks.

- Again, $\leq\, = \,\leq_K \circ \leq_{P/K} \circ \leq_P$, on $\mathrm{Sub}(G_1 \times G_2)$.

Theorem (Masterson-Pf. 2017)

$$M(G_1 \times G_2) = \mathbf{D}_0 \cdot \mathbf{A}(\leq_K) \cdot \mathbf{A}(\leq_{P/K}) \cdot \mathbf{A}(\leq_P),$$

where:

- $\mathbf{A}(\leq_K) = \bigoplus_{[K_1]} \bigoplus_{[K_2]} \mathbf{A}_{K_1,K_2}(\leq)$,
  $N_{G_1}(K_1) \times N_{G_2}(K_2) \curvearrowright \{L \in \mathrm{Sub}(G_1 \times G_2) : k_i(L) = K_i\}$;
- $\mathbf{A}(\leq_P) = \bigoplus_{[P_1]} \bigoplus_{[P_2]} \mathbf{A}_{P_1,P_2}(\leq)$,
  $N_{G_1}(P_1) \times N_{G_2}(P_2) \curvearrowright \{L \in \mathrm{Sub}(G_1 \times G_2) : p_i(L) = P_i\}$;
- $\mathbf{A}(\leq_{P/K}) = \bigoplus_{U} \mathbf{A}_U^{G_1}(\leq) \otimes_{\mathrm{Aut}U} \mathbf{A}_U^{G_2}(\leq)$, $G_i \curvearrowright \mathrm{Mor}_U(G_i)$.

Table of Marks.

- Again, $\leq \, = \, \leq_K \circ \leq_{P/K} \circ \leq_P$, on $\mathrm{Sub}(G_1 \times G_2)$.

Theorem (Masterson-Pf. 2017)

$$M(G_1 \times G_2) = \mathbf{D}_0 \cdot \mathbf{A}(\leq_K) \cdot \mathbf{A}(\leq_{P/K}) \cdot \mathbf{A}(\leq_P),$$

where:

- $\mathbf{A}(\leq_K) = \bigoplus_{[K_1]} \bigoplus_{[K_2]} \mathbf{A}_{K_1,K_2}(\leq)$,
  $N_{G_1}(K_1) \times N_{G_2}(K_2) \curvearrowright \{L \in \mathrm{Sub}(G_1 \times G_2) : k_i(L) = K_i\}$;
- $\mathbf{A}(\leq_P) = \bigoplus_{[P_1]} \bigoplus_{[P_2]} \mathbf{A}_{P_1,P_2}(\leq)$,
  $N_{G_1}(P_1) \times N_{G_2}(P_2) \curvearrowright \{L \in \mathrm{Sub}(G_1 \times G_2) : p_i(L) = P_i\}$;
- $\mathbf{A}(\leq_{P/K}) = \bigoplus_{U} \mathbf{A}_U^{G_1}(\leq) \otimes_{\mathrm{Aut} U} \mathbf{A}_U^{G_2}(\leq)$, $G_i \curvearrowright \mathrm{Mor}_U(G_i)$.

Table of Marks.

- Again, $\leq \, = \, \leq_K \circ \leq_{P/K} \circ \leq_P$, on $\mathrm{Sub}(G_1 \times G_2)$.

Theorem (Masterson-Pf. 2017)

$$M(G_1 \times G_2) = \mathbf{D}_0 \cdot \mathbf{A}(\leq_K) \cdot \mathbf{A}(\leq_{P/K}) \cdot \mathbf{A}(\leq_P),$$

where:

- $\mathbf{A}(\leq_K) = \bigoplus_{[K_1]} \bigoplus_{[K_2]} \mathbf{A}_{K_1,K_2}(\leq)$,
  $N_{G_1}(K_1) \times N_{G_2}(K_2) \curvearrowright \{L \in \mathrm{Sub}(G_1 \times G_2) : k_i(L) = K_i\}$;
- $\mathbf{A}(\leq_P) = \bigoplus_{[P_1]} \bigoplus_{[P_2]} \mathbf{A}_{P_1,P_2}(\leq)$,
  $N_{G_1}(P_1) \times N_{G_2}(P_2) \curvearrowright \{L \in \mathrm{Sub}(G_1 \times G_2) : p_i(L) = P_i\}$;
- $\mathbf{A}(\leq_{P/K}) = \bigoplus_{U} \mathbf{A}_U^{G_1}(\leq) \otimes_{\mathrm{Aut}U} \mathbf{A}_U^{G_2}(\leq)$, $G_i \curvearrowright \mathrm{Mor}_U(G_i)$.

Sections Lattice Revisited.

Note

- $(P', K') \leq (P, K)$ does not imply $|P'/K'| \leq |P/K|$.

- Define a relation

$$\leq' := \leq_K \circ \leq_u \circ \geq_P.$$

Theorem

- $\leq'$ *is a partial order on* $\mathrm{Sec}(G)$.
- $(P', K') \leq' (P, K) \iff P' \leq P$ *and* $K \cap P' \leq K'$.
- $\mathbf{A}(\leq') = \mathbf{A}(\leq_K) \cdot \mathbf{A}(\leq_{P/K}) \cdot \mathbf{A}(\geq_P)$.

- $(P', K') \leq' (P, K)$ **does** imply $|P'/K'| \leq |P/K| \ldots$

Sections Lattice Revisited.

Note

- $(P', K') \leq (P, K)$ does not imply $|P'/K'| \leq |P/K|$.

- Define a relation

$$\leq' := \leq_K \circ \leq_u \circ \geq_P.$$

Theorem

- *$\leq'$ is a partial order on* $\mathrm{Sec}(G)$.
- $(P', K') \leq' (P, K) \iff P' \leq P$ *and* $K \cap P' \leq K'$.
- $\mathbf{A}(\leq') = \mathbf{A}(\leq_K) \cdot \mathbf{A}(\leq_{P/K}) \cdot \mathbf{A}(\geq_P)$.

- $(P', K') \leq' (P, K)$ **does** imply $|P'/K'| \leq |P/K| \ldots$

Sections Lattice Revisited.

Note

- $(P', K') \leq (P, K)$ does not imply $|P'/K'| \leq |P/K|$.

- Define a relation

$$\leq' := \leq_K \circ \leq_u \circ \geq_P.$$

Theorem

- $\leq'$ *is a partial order on* $\mathrm{Sec}(G)$.
- $(P', K') \leq' (P, K) \iff P' \leq P$ *and* $K \cap P' \leq K'$.
- $A(\leq') = A(\leq_K) \cdot A(\leq_{P/K}) \cdot A(\geq_P)$.

- $(P', K') \leq' (P, K)$ **does** imply $|P'/K'| \leq |P/K| \ldots$

Sections Lattice Revisited.

Note

- $(P', K') \leq (P, K)$ does not imply $|P'/K'| \leq |P/K|$.

- Define a relation

$$\leq' := \leq_K \circ \leq_u \circ \geq_P.$$

Theorem

- *$\leq'$ is a partial order on $\mathrm{Sec}(G)$.*
- *$(P', K') \leq' (P, K) \iff P' \leq P$ and $K \cap P' \leq K'$.*
- *$\mathbf{A}(\leq') = \mathbf{A}(\leq_K) \cdot \mathbf{A}(\leq_{P/K}) \cdot \mathbf{A}(\geq_P)$.*

- $(P', K') \leq' (P, K)$ **does** imply $|P'/K'| \leq |P/K| \ldots$

Sections Lattice Revisited.

Note

- $(P', K') \leq (P, K)$ does not imply $|P'/K'| \leq |P/K|$.

- Define a relation

$$\leq' := \leq_K \circ \leq_u \circ \geq_P.$$

Theorem

- $\leq'$ *is a partial order on* $\mathrm{Sec}(G)$.
- $(P', K') \leq' (P, K) \iff P' \leq P$ *and* $K \cap P' \leq K'$.
- $\mathbf{A}(\leq') = \mathbf{A}(\leq_K) \cdot \mathbf{A}(\leq_{P/K}) \cdot \mathbf{A}(\geq_P)$.

- $(P', K') \leq' (P, K)$ **does** imply $|P'/K'| \leq |P/K| \ldots$

Sections Lattice Revisited.

Note

- $(P', K') \leq (P, K)$ does not imply $|P'/K'| \leq |P/K|$.

- Define a relation

$$\leq' := \leq_K \circ \leq_u \circ \geq_P.$$

Theorem

- $\leq'$ *is a partial order on* $\mathrm{Sec}(G)$.
- $(P', K') \leq' (P, K) \iff P' \leq P$ *and* $K \cap P' \leq K'$.
- $\mathbf{A}(\leq') = \mathbf{A}(\leq_K) \cdot \mathbf{A}(\leq_{P/K}) \cdot \mathbf{A}(\geq_P)$.

- $(P', K') \leq' (P, K)$ **does** imply $|P'/K'| \leq |P/K| \ldots$

Sections Lattice Revisited.

Note

- $(P', K') \leq (P, K)$ does not imply $|P'/K'| \leq |P/K|$.

- Define a relation

$$\leq' := \leq_K \circ \leq_u \circ \geq_P.$$

Theorem

- $\leq'$ *is a partial order on* $\mathrm{Sec}(G)$.
- $(P', K') \leq' (P, K) \iff P' \leq P$ *and* $K \cap P' \leq K'$.
- $\mathbf{A}(\leq') = \mathbf{A}(\leq_K) \cdot \mathbf{A}(\leq_{P/K}) \cdot \mathbf{A}(\geq_P)$.

- $(P', K') \leq' (P, K)$ **does** imply $|P'/K'| \leq |P/K| \ldots$

Sections Lattice Revisited.

Note

- $(P', K') \leq (P, K)$ does not imply $|P'/K'| \leq |P/K|$.

- Define a relation

$$\leq' := \leq_K \circ \leq_u \circ \geq_P.$$

Theorem

- $\leq'$ *is a partial order on* $\mathrm{Sec}(G)$.
- $(P', K') \leq' (P, K) \iff P' \leq P$ *and* $K \cap P' \leq K'$.
- $\mathbf{A}(\leq') = \mathbf{A}(\leq_K) \cdot \mathbf{A}(\leq_{P/K}) \cdot \mathbf{A}(\geq_P)$.

- $(P', K') \leq' (P, K)$ **does** imply $|P'/K'| \leq |P/K| \ldots$

The Double Burnside Ring of Sym_3.

- Let $G = \mathrm{Sym}_3$. Then $\mathbb{Q}B(G, G)$ has a basis $\{b_1, b_2, \ldots, b_{22}\}$ of transitive $G \times G$-sets, labelled by the conjugacy classes of subgroups of $G \times G$.
- Use Bouc's Mackey formula to compute matrices for the (right) regular representation of $\mathbb{Q}(B, B)$ in terms of the b_i.
- Set

$$M' = \mathbf{D}_0 \cdot \mathbf{A}(\geq_K) \cdot \mathbf{A}(\geq_{P/K}) \cdot \mathbf{D}_1 \cdot \mathbf{A}(\leq_P) \cdot \mathbf{D}_2$$

  for certain diagonal matrices $\mathbf{D}_1$ and $\mathbf{D}_2$.
- Then define elements $c_j \in \mathbb{Q}B(G, G)$ by the equations

$$b_i = \textstyle\sum_j m'_{ij} c_j.$$

- Computing the regular representation in terms of the c_j then shows that $\mathbb{Q}B(G, G)$ is quasi-hereditary with a cellular structure as follows:

The Double Burnside Ring of Sym_3.

- Let $G = \mathrm{Sym}_3$. Then $\mathbb{Q}B(G, G)$ has a basis $\{b_1, b_2, \ldots, b_{22}\}$ of transitive $G \times G$-sets, labelled by the conjugacy classes of subgroups of $G \times G$.
- Use Bouc's Mackey formula to compute matrices for the (right) regular representation of $\mathbb{Q}(B, B)$ in terms of the b_i.
- Set

$$M' = \mathbf{D}_0 \cdot \mathbf{A}(\geq_K) \cdot \mathbf{A}(\geq_{P/K}) \cdot \mathbf{D}_1 \cdot \mathbf{A}(\leq_P) \cdot \mathbf{D}_2$$

  for certain diagonal matrices $\mathbf{D}_1$ and $\mathbf{D}_2$.
- Then define elements $c_j \in \mathbb{Q}B(G, G)$ by the equations

$$b_i = \sum_j m'_{ij} c_j.$$

- Computing the regular representation in terms of the c_j then shows that $\mathbb{Q}B(G, G)$ is quasi-hereditary with a cellular structure as follows:

The Double Burnside Ring of Sym_3.

- Let $G = \mathrm{Sym}_3$. Then $\mathbb{Q}B(G, G)$ has a basis $\{b_1, b_2, \dots, b_{22}\}$ of transitive $G \times G$-sets, labelled by the conjugacy classes of subgroups of $G \times G$.
- Use Bouc's Mackey formula to compute matrices for the (right) regular representation of $\mathbb{Q}(B, B)$ in terms of the b_i.
- Set
$$M' = \mathbf{D}_0 \cdot \mathbf{A}(\geq_K) \cdot \mathbf{A}(\geq_{P/K}) \cdot \mathbf{D}_1 \cdot \mathbf{A}(\leq_P) \cdot \mathbf{D}_2$$
for certain diagonal matrices $\mathbf{D}_1$ and $\mathbf{D}_2$.
- Then define elements $c_j \in \mathbb{Q}B(G, G)$ by the equations
$$b_i = \sum_j m'_{ij} c_j.$$
- Computing the regular representation in terms of the c_j then shows that $\mathbb{Q}B(G, G)$ is quasi-hereditary with a cellular structure as follows:

The Double Burnside Ring of Sym_3.

- Let $G = \mathrm{Sym}_3$. Then $\mathbb{Q}B(G, G)$ has a basis $\{b_1, b_2, \ldots, b_{22}\}$ of transitive $G \times G$-sets, labelled by the conjugacy classes of subgroups of $G \times G$.
- Use Bouc's Mackey formula to compute matrices for the (right) regular representation of $\mathbb{Q}(B, B)$ in terms of the b_i.
- Set
$$M' = \mathbf{D}_0 \cdot \mathbf{A}(\geq_K) \cdot \mathbf{A}(\geq_{P/K}) \cdot \mathbf{D}_1 \cdot \mathbf{A}(\leq_P) \cdot \mathbf{D}_2$$
for certain diagonal matrices $\mathbf{D}_1$ and $\mathbf{D}_2$.
- Then define elements $c_j \in \mathbb{Q}B(G, G)$ by the equations
$$b_i = \textstyle\sum_j m'_{ij} c_j.$$
- Computing the regular representation in terms of the c_j then shows that $\mathbb{Q}B(G, G)$ is quasi-hereditary with a cellular structure as follows:

The Double Burnside Ring of Sym_3.

- Let $G = \mathrm{Sym}_3$. Then $\mathbb{Q}B(G, G)$ has a basis $\{b_1, b_2, \ldots, b_{22}\}$ of transitive $G \times G$-sets, labelled by the conjugacy classes of subgroups of $G \times G$.
- Use Bouc's Mackey formula to compute matrices for the (right) regular representation of $\mathbb{Q}(B, B)$ in terms of the b_i.
- Set
$$M' = \mathbf{D}_0 \cdot \mathbf{A}(\geq_K) \cdot \mathbf{A}(\geq_{P/K}) \cdot \mathbf{D}_1 \cdot \mathbf{A}(\leq_P) \cdot \mathbf{D}_2$$
for certain diagonal matrices $\mathbf{D}_1$ and $\mathbf{D}_2$.
- Then define elements $c_j \in \mathbb{Q}B(G, G)$ by the equations
$$b_i = \textstyle\sum_j m'_{ij} c_j.$$
- Computing the regular representation in terms of the c_j then shows that $\mathbb{Q}B(G, G)$ is quasi-hereditary with a cellular structure as follows:

The Double Burnside Ring of Sym_3, cont'd.

Theorem

Let $G = \mathrm{Sym}_3$. *Then the map* $\beta'_{G\times G} \colon \mathbb{Q}B(G,G) \to \mathbb{Q}^{8\times 8}$ *defined by*

$$\beta'_{G\times G}\Big(\sum_i x_i c_i\Big) = \left(\begin{array}{cccc|cc|c|c}
x_1 & x_2 & x_3 & x_4 & . & . & . & . \\
x_5 & x_6 & x_7 & x_8 & . & . & . & . \\
x_9 & x_{10} & x_{11} & x_{12} & . & . & . & . \\
. & . & . & x_{22} & . & . & . & . \\
\hline
. & . & . & . & x_{17} & x_{18} & . & . \\
. & . & . & . & . & x_{22} & . & . \\
\hline
. & . & . & . & . & . & x_{21} & . \\
\hline
x_{13} & x_{14} & x_{15} & x_{16} & x_{19} & x_{20} & . & x_{22}
\end{array}\right),$$

$x_i \in \mathbb{Q}$, *is an injective homomorphism of algebras.*

Further Results and Open Questions.

- Does $B(G, G)$ always have a cellular structure?
- In general, the situation is more complicated, and computing examples is a challenge as direct products have many (classes of) subgroups ...
- $G = C_n$ cylic: $\mathbb{Q}B(G, G)$ is semisimple (Boltje-Danz).
- $G = D_n$ dihedral, n squarefree, is essentially like $G = \mathrm{Sym}_3$ (jt. with S. Park).
- Further examples: 2^2, A_4, A_5.
- $G = A_5$: $\mathbb{Q}B(G, G)$ has infinite global dimension (Rognerut).
- $G = D_8$?

Further Results and Open Questions.

- Does $B(G, G)$ always have a cellular structure?
- In general, the situation is more complicated, and computing examples is a challenge as direct products have many (classes of) subgroups ...
- $G = C_n$ cylic: $\mathbb{Q}B(G, G)$ is semisimple (Boltje-Danz).
- $G = D_n$ dihedral, n squarefree, is essentially like $G = \mathrm{Sym}_3$ (jt. with S. Park).
- Further examples: 2^2, A_4, A_5.
- $G = A_5$: $\mathbb{Q}B(G, G)$ has infinite global dimension (Rognerut).
- $G = D_8$?

Further Results and Open Questions.

- Does $B(G, G)$ always have a cellular structure?
- In general, the situation is more complicated, and computing examples is a challenge as direct products have many (classes of) subgroups ...
- $G = C_n$ cylic: $\mathbb{Q}B(G, G)$ is semisimple (Boltje-Danz).
- $G = D_n$ dihedral, n squarefree, is essentially like $G = \mathrm{Sym}_3$ (jt. with S. Park).
- Further examples: 2^2, A_4, A_5.
- $G = A_5$: $\mathbb{Q}B(G, G)$ has infinite global dimension (Rognerut).
- $G = D_8$?

Further Results and Open Questions.

- Does $B(G, G)$ always have a cellular structure?
- In general, the situation is more complicated, and computing examples is a challenge as direct products have many (classes of) subgroups ...
- $G = C_n$ cylic: $\mathbb{Q}B(G, G)$ is semisimple (Boltje-Danz).
- $G = D_n$ dihedral, n squarefree, is essentially like $G = \mathrm{Sym}_3$ (jt. with S. Park).
- Further examples: 2^2, A_4, A_5.
- $G = A_5$: $\mathbb{Q}B(G, G)$ has infinite global dimension (Rognerut).
- $G = D_8$?

Further Results and Open Questions.

- Does $B(G, G)$ always have a cellular structure?
- In general, the situation is more complicated, and computing examples is a challenge as direct products have many (classes of) subgroups ...
- $G = C_n$ cylic: $\mathbb{Q}B(G, G)$ is semisimple (Boltje-Danz).
- $G = D_n$ dihedral, n squarefree, is essentially like $G = \mathrm{Sym}_3$ (jt. with S. Park).
- Further examples: 2^2, A_4, A_5.
- $G = A_5$: $\mathbb{Q}B(G, G)$ has infinite global dimension (Rognerut).
- $G = D_8$?

Further Results and Open Questions.

- Does $B(G, G)$ always have a cellular structure?
- In general, the situation is more complicated, and computing examples is a challenge as direct products have many (classes of) subgroups . . .
- $G = C_n$ cylic: $\mathbb{Q}B(G, G)$ is semisimple (Boltje-Danz).
- $G = D_n$ dihedral, n squarefree, is essentially like $G = \mathrm{Sym}_3$ (jt. with S. Park).
- Further examples: 2^2, A_4, A_5.
- $G = A_5$: $\mathbb{Q}B(G, G)$ has infinite global dimension (Rognerut).
- $G = D_8$?

Further Results and Open Questions.

- Does $B(G, G)$ always have a cellular structure?
- In general, the situation is more complicated, and computing examples is a challenge as direct products have many (classes of) subgroups ...
- $G = C_n$ cylic: $\mathbb{Q}B(G, G)$ is semisimple (Boltje-Danz).
- $G = D_n$ dihedral, n squarefree, is essentially like $G = \mathrm{Sym}_3$ (jt. with S. Park).
- Further examples: 2^2, A_4, A_5.
- $G = A_5$: $\mathbb{Q}B(G, G)$ has infinite global dimension (Rognerut).
- $G = D_8$?