Primitive permutation groups and generalised quadrangles

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Generalised quadrangle (GQ): point–line geometry $\mathcal{Q}$ such that

(i) two distinct points are incident with at most one common line;

(ii) if a point and line are not incident, they are joined by a unique line.

Example 1 Take points, lines to be the totally isotropic 1, 2 spaces w.r.t. a nondegenerate alternating form on $\mathbb{F}_q^4$, with natural incidence. Then $\mathcal{Q}$ is a GQ with $\mathrm{Aut}(\mathcal{Q}) = \mathrm{P\Gamma Sp}(4, q)$ acting primitively on points and lines, and transitively on flags (incident point–line pairs).

Example 2 Other "classical" examples, admitting (overgroups of) $\mathrm{PSU}(4, q)$ or $\mathrm{PSU}(5, q)$. Also point- and line-primitive, flag-transitive.

Example 3 Various 'synthetic' constructions (due to Payne, Tits, Ahrens–Szekeres, Hall . . .). Typically point- and/or line-intransitive.

Conjecture (e.g. Kantor, 1990) The only flag-transitive GQs are the classical families and two (known, small) examples with affine groups.

All these examples are also point-primitive (up to duality) so one might seek to classify the point-primitive GQs.

Theorem (Bamberg et. al, 2012) If $G \leqslant \mathrm{Aut}(\mathcal{Q})$ is point- and line-primitive and flag-transitive, then G is almost simple of Lie type.

Theorem (BPP & Glasby, 2016) If G is affine, point-primitive, line-transitive then $\mathcal{Q}$ is one of the two examples in the conjecture.

Theorem (BPP, 2017) If G is point-primitive, line-transitive then its O'Nan–Scott type is not HS or HC.

Suppose now that $G \leqslant \mathrm{Aut}(\mathcal{Q})$ acts primitively on points.

The affine case seems hard without line-transitivity, amounting to a classification of certain "hyperovals" in $\mathrm{PG}(2, 2^f)$.

For non-affine G, we can say a lot without assuming line-transitivity, by considering the *fixity* of the point action.

Theorem (BPP, 2017+) Let $\theta \neq 1$ be any automorphism of any $\mathcal{Q}$. Then either θ fixes less than $|\mathcal{P}|^{4/5}$ points, or $\mathcal{Q}$ is the unique GQ of order $(2, 4)$ and θ fixes exactly 15 of the 27 points of $\mathcal{Q}$.

Remark: Babai (2015) shows that an automorphism of a strongly regular graph on ℓ vertices can fix at most $O(\ell^{7/8})$ vertices, but the improvement $7/8 \to 4/5$ in our special case turns out to be useful.

O'Nan–Scott types HS, HC, SD, CD Here for some non-abelian finite simple group T and $k \geqslant 2$, $H = T^k$ acts on $\Omega = T^{k-1} \times \{1\} \leqslant H$ via

$$(y_1, \ldots, y_{k-1}, 1)^{(x_1, \ldots, x_{k-1}, x_k)} = (x_k^{-1} y_1 x_1, \ldots, x_k^{-1} y_{k-1} x_{k-1}, 1),$$

we can identify $\mathcal{P} = \Omega^r$ for some $r \geqslant 1$, and G has a subgroup $N = H^r$ with product action on $\mathcal{P}$. (Type HS, HC: $k = 2$. Type SD, CD: $k > 2$.)

Lemma If $G \leqslant \mathrm{Aut}(\mathcal{Q})$ is as above, then $r \leqslant 3$.

Proof. Choose $x \in T$ with 'large' centraliser, say $|C_T(x)| > |T|^{1/3}$. Then $\hat{x} := (x, \ldots, x) \in T^k = H$ fixes $(y_1, \ldots, y_{k-1}, 1) \in T^{k-1} = \Omega$ iff $y_1, \ldots, y_{k-1} \in C_T(x)$, so $\theta = (\hat{x}, 1, \ldots, 1) \in H^r = N \leqslant G \leqslant \mathrm{Aut}(\mathcal{Q})$ fixes $|C_T(x)|^{k-1} |T^{k-1}|^{r-1} > (|T|^{k-1})^{r-1+1/3}$ elements of $\mathcal{P} = \Omega^r$. By our theorem, θ cannot fix more than $|\mathcal{P}|^{4/5} = |T^{(k-1)}|^{4r/5}$ points, so it follows that $r - 1 + 1/3 < 4r/5$, and hence $r \leqslant 3$. $\square$

We can usually do better by choosing x with larger centraliser, and for $k = 2$ we can also restrict the involution structure of T.

We are able to conclude that type HC does not arise, and the following:

Type	T must be one of the following
HS	Lie type $\mathrm{A}_5^{\pm}$, $\mathrm{A}_6^{\pm}$, B_3, C_2, C_3, $\mathrm{D}_4^{\pm}$, $\mathrm{D}_5^{\pm}$, $\mathrm{D}_6^{\pm}$, $\mathrm{E}_6^{\pm}$, E_7 or F_4
SD	sporadic, or Alt_n with $n \leqslant 18$, or exceptional Lie type, or type $\mathrm{A}_n^{\pm}$ or $\mathrm{D}_n^{\pm}$ with $n \leqslant 8$, or type B_n or C_n with $n \leqslant 4$
CD, $r = 2$	sporadic (with six exceptions), or Alt_n with $n \leqslant 9$, or Lie type A_1, $\mathrm{A}_2^{\pm}$, $\mathrm{A}_3^{\pm}$, B_2, ${}^2\mathrm{B}_2$, ${}^2\mathrm{F}_4$, G_2 or ${}^2\mathrm{G}_2$
CD, $r = 3$	J_1, or Lie type A_1 or ${}^2\mathrm{B}_2$

It should be possible to complete type HS using the involution structure of the remaining candidates for T. SD and CD seem harder to finish.

O'Nan–Scott type PA Here $T^r \leqslant G \leqslant H \wr \mathrm{Sym}_r$ for some almost simple primitive group $H \leqslant \mathrm{Sym}(\Omega)$ and some $r \geqslant 2$ (or $r = 1$ when G is AS). Our 'fixity theorem' implies that every non-identity element of H must fix less than $|\Omega|^{1-r/5}$ elements of Ω, and in particular that $r \leqslant 4$.

Most primitive actions H^Ω have $f(H) \geqslant |\Omega|^{4/9}$, realised by an involution. The exceptions are classified by Covato (classical, alternating, sporadic groups) and Burness–Thomas (exceptional groups).

Since $4/9 > 1 - 3/5 = 2/5$, we can use this to restrict the possibilities for H^Ω when $r \in \{3, 4\}$, e.g. if T is alternating or sporadic then $H = T \cong \mathrm{Alt}_p$ for $p \equiv 3 \pmod 4$ a prime, with point stabiliser $p.\frac{p-1}{2}$.

The $4/9$ exponent can sometimes be improved, but is best possible in infinitely many cases. Improving to $3/5$ would leave a list of exceptions for $r = 2$. (Moreover, we don't need involutions for our application.)

O’Nan–Scott type TW The twisted wreath product case seems hard.

Here $G = N \rtimes P$ with $N \cong T^r$ acting regularly by right multiplication and $P \leqslant \mathrm{Sym}_r$ acting by conjugation and permuting the factors of N transitively (plus some other, more complicated conditions).

The regular subgroup doesn’t seem to help much, beyond imposing the Diophantine equation $|T|^r = |\mathcal{P}| = (s+1)(st+1)$ subject to the constraints $2^{1/2} \leqslant s^{1/2} \leqslant t \leqslant s^2 \leqslant t^4$ and $s+t \mid st(st+1)$, where (s,t) is the *order* of $\mathcal{Q}$ (lines have $s+1$ points, points are on $t+1$ lines).

Moreover, there seems to be no sufficiently strong fixity bound to put into our theorem: Liebeck and Shalev (2015) deduce a $|T^r|^{1/3}$ lower bound, which can sometimes be improved to $|T^r|^{1/2}$, but this far away from the $|T^r|^{4/5}$ upper bound imposed by the theorem.

Thank you!

Open problem 1 When can the point set of a (thick, finite) GQ have size $|T|^r$ for some non-abelian finite simple group T and some $r \geqslant 1$?

Open problem 2 Which almost simple primitive groups $H \leqslant \mathrm{Sym}(\Omega)$ have fixity $> |\Omega|^{3/5}$? What about $|\Omega|^{4/5}$? (The elements realising these bounds need not be involutions; any non-identity element will do.)

Open problem 3 When does a primitive group $G \leqslant \mathrm{Sym}(\Omega)$ of TW type have fixity $> |\Omega|^{4/5}$, or something 'close' to this, e.g. $|\Omega|^{3/4}$?