q-Tensor Squares of Polycyclic Groups

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(This is a joint work with E. Rodrigues and I. Dias, Brazil)

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Let G be any group and G^{φ} an isomorphic copy of G, where $g \mapsto g^{\varphi}$, $\forall g \in G$. Define

$$\nu(G) := \langle G, G^{\varphi} \mid [g_1, g_2{}^{\varphi}]^{g_3} = [g_1{}^{g_3}, (g_2{}^{g_3})^{\varphi}] = [g_1, g_2{}^{\varphi}]^{g_3{}^{\varphi}} \rangle,$$

for all $g_1, g_2, g_3 \in G$. (N.R., 1991)

Well known fact:

$\Upsilon(G) := [G, G^{\varphi}] \cong G \otimes G$, the non-abelian tensor square of G.

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Extension to a modular version

D.Conduché & C.Rodriguez-Fernandez (1992): non-abelian tensor product of q-crossed modules, $q \geq 0$.

Relations are abstractions of ($q^{th}-$) powers and commutator relations in groups.

R. Brown (1990) explores its relationship with universal q-central extensions of q-perfect groups.

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The Group $\nu^q(G)$

T. Bueno (2006) introduces a variant of Ellis construction by extending the commutator approach from $\nu(G)$ to a *hat and commutator* approach, $\nu^q(G)$, for all $q \geq 0$.

For $q \geq 1$ let $\widehat{\mathcal{G}} = \{\widehat{k} \mid k \in G\}$ be a set of symbols and let $F(\widehat{\mathcal{G}})$ be the free group over $\widehat{\mathcal{G}}$ (for $q = 0$ set $\widehat{\mathcal{G}} = \emptyset$).

In the free product $\nu(G) * F(\widehat{\mathcal{G}})$, let J be the normal closure of the following elements, for all $g, h, k, k_1 \in G$:

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Defining Relations

$$g^{-1}\,\widehat{k}\,g\,(\widehat{k^g})^{-1}; \tag{1}$$

$$(h^{\varphi})^{-1}\,\widehat{k}\,h^{\varphi}\,(\widehat{k^h})^{-1}; \tag{2}$$

$$(\widehat{k})^{-1}[g,h^{\varphi}]\,\widehat{k}\,[g^{k^q},(h^{k^q})^{\varphi}]^{-1}; \tag{3}$$

$$(\widehat{k})^{-1}\,\widehat{kk_1}\,(\widehat{k_1})^{-1}(\prod_{i=1}^{q-1}[k,(k_1^{-i})^{\varphi}]^{k^{q-1-i}})^{-1}; \tag{4}$$

$$[\widehat{k},\widehat{k_1}]\,[k^q,(k_1^q)^{\varphi}]^{-1}; \tag{5}$$

$$\widehat{[g,h]}\,[g,h^{\varphi}]^{-q}. \tag{6}$$

$\nu^q(G)$ and some relevant sections

$$\boxed{\nu^q(G) := (\nu(G) * F(\widehat{\mathcal{G}}))/J}$$

For $q = 0$ the set of all relations (1) to (6) is empty;
in this case $\nu(G) * F(\widehat{\mathcal{G}})/J \cong \nu(G)$

G and G^φ are embedded into $\nu^q(G)$, for all $q \geq 0$

Set $T = [G, G^\varphi] \leq \nu^q(G)$ and let $\mathfrak{G}$ be the subgroup of $\nu^q(G)$ generated by (the images of) $\widehat{\mathcal{G}}$

The subgroup $\Upsilon^q(G) = T\mathfrak{G}$ is normal in $\nu^q(G)$ and
$\nu^q(G) = G^\varphi \cdot (G \cdot \Upsilon^q(G))$

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The *q-exterior square* $G \wedge^q G$, is the quotient of $G \otimes^q G$ by its subgroup $\langle k \otimes k | k \in G \rangle$. Thus,

$G \wedge^q G \cong \Upsilon^q(G)/\Delta^q(G)$, where $\Delta^q(G) = \langle [g, g^\varphi] \mid g \in G \rangle$.

Let $\rho' : \Upsilon^q(G) \to G$ be induced by $[g, h^\varphi] \mapsto [g, h]$, $\widehat{k} \mapsto k^q$, for all $g, h, k \in G$.

$\Rightarrow \boxed{\mathrm{Ker}(\rho')/\Delta^q(G) \cong H_2(G, \mathbb{Z}_q).}$

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T.Bueno & N.R. [2011]

Let $d = \gcd(q, n)$. Then

$$C_\infty \otimes^q C_\infty \cong C_\infty \times C_q,$$

$$C_n \otimes^q C_n \cong \begin{cases} C_n \times C_d, & \text{if} \quad d \text{ is odd,} \\ C_n \times C_d, & \text{if} \quad d \text{ is even and either } 4|n \text{ or } 4|q; \\ C_{2n} \times C_{d/2}, & \text{otherwise.} \end{cases}$$

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Let $G = N \times H$, $\overline{N} = N/N'N^q$ and $\overline{H} = H/H'H^q$. Then

(i) $\nu^q(G) = \langle N, N^\varphi, \widehat{\mathcal{N}} \rangle \times [N, H^\varphi][H, N^\varphi] \times \langle H, H^\varphi, \widehat{\mathcal{H}} \rangle$;

(ii) $\langle H, H^\varphi, \widehat{\mathcal{H}} \rangle \cong \nu^q(H)$; $\langle N, N^\varphi, \widehat{\mathcal{N}} \rangle \cong \nu^q(N)$.

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(iv) $[N, H^\varphi] \cong \overline{N} \otimes_{\mathbb{Z}_q} \overline{H} \cong [H, N^\varphi]$.

(v) $\Delta^q(N \times H) = \Delta^q(N) \times \Delta(H) \times U$, where

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f.g. Abelian Groups

If $A = \prod_{i=1}^{r} C_i$, where $C_i = \langle x_i \rangle$, then

$$\Upsilon^q(A) = \prod_{i=1}^{r} \Upsilon^q(C_i) \times \prod_{1 \leq j < k \leq r} [C_j, C_k^{\varphi}][C_k, C_j^{\varphi}].$$

Here we have $\Upsilon^q(C_i) = \langle [x_i, x_i^{\varphi}],\ \widehat{x}_i \rangle$ and
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$[C_j, C_k^{\varphi}][C_k, C_j^{\varphi}] = \langle\, [x_j, x_k^{\varphi}][x_k, x_j^{\varphi}],\ [x_j, x_k^{\varphi}]\,\rangle$.
In addition, $\Delta^q(A) = \prod_{i=1}^{r} \langle [x_i, x_i^{\varphi}] \rangle \times \prod_{j<k} \langle [x_j, x_k^{\varphi}][x_k, x_j^{\varphi}] \rangle$.

Write $E_X^q(A) := \langle \widehat{x_i},\, [x_j, x_k^{\varphi}] \mid 1 \le i \le r,\, 1 \le j < k \le r \rangle$.

We get $\Upsilon^q(A) = \Delta^q(A) E_X^q(A)$.

For $q = 0$, we have $\Delta(A) \cap E_X(A) = 1$, giving the direct decomposition:

$$\Upsilon(A) = \Delta(A) \times E_X(A).$$

If $q > 0$ the above decomposition is not in general valid. In fact, $\Upsilon^q(G) = T\mathfrak{G}$ and $\Delta^q(G) \le T$, for any group G; thus, such a decompositions will depend on the relationship between T and $\mathfrak{G}$.

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Remark on $\Delta^q(G)$

$\Delta^q(G)$ essentially depends on q and $G^{ab} = G/G'$

$\Delta^q(G)$ does not depend on the particular set of generators of G:

[N.R., 1994]

Let $X = \{x_i\}_{i \in I}$ be a set of generators of grp G (assume I totally ordered). Then $\Delta^q(G)$ is generated by the set

$$\Delta_X = \{s_i := [x_i, x_i^{\varphi}],\ t_{jk} := [x_j, x_k^{\varphi}][x_k, x_j^{\varphi}],\ |\ i, j, k \in I, j < k\}.$$

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For $q = 0$, [Blyth, Fumagalli and Morigi, 2009] extending results of [Brown, Johnson & Robertson, 1987] and of [N.R., 1991, 1994] established conditions on f.g. G^{ab} in order to describe the structure of $G \otimes G$ as a direct product of $\Delta(G)$ and the exterior square $G \wedge G$:

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Let G be a group s.t. G^{ab} is f.g. If G^{ab} has no element of order 2, or if G' has a complement in G, the $G \otimes G \cong \Delta(G) \times (G \wedge G)$

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Let $q > 1$ be an odd integer and A a finitely generated abelian group given by the presentation

$$A = \langle x_1, \ldots, x_r \mid x_i^{n_i}, [x_j, x_k], 1 \leq i, j, k \leq r, j < k \rangle,$$

where we assume that $n_{l+1} = \cdots = n_r = 0$ in case the free part of A is generated by $\{x_{l+1}, \ldots, x_r\}$, $0 \leq l \leq r - 1$. Write $d_i = \gcd(q, n_i)$ and $d_{jk} = \gcd(q, n_j, n_k)$, $1 \leq i, j, k \leq r, j < k$. Then

(i) $\Delta^q(A) \cong \prod_{i=1}^{r} C_{d_i} \times \prod_{1 \leq j < k \leq r} C_{d_{jk}}$ and $E_X^q(A) \cong A \times \prod_{1 \leq j < k \leq r} C_{d_{jk}}$;

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[G. Ellis, 1989] - Computing $G \wedge^q G$ from a presentation of G

Let $q \geq 0$ and let F/R be a free presentation of G. Then

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[Corollary]

Let F_n be the free group of rank n. Then

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Let $\mathcal{N}_{n,2}$ be the free nilpotent group of rank $n > 1$ and class 2, $\mathcal{N}_{n,2} = F_n/\gamma_3(F_n)$. Then,

(i) [M.Bacon, 1994] $\mathcal{N}_{n,2} \otimes \mathcal{N}_{n,2}$ is free abelian of rank $\frac{1}{3}n(n^2 + 3n - 1)$.
More precisely, we have
$\mathcal{N}_{n,2} \otimes \mathcal{N}_{n,2} \cong \Delta(F_n^{ab}) \times M(\mathcal{N}_{n,2}) \times \mathcal{N}'_{n,2}$.

(ii) [R. R., 2016] For $q > 1$ and q odd,
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f.g. nilpotent groups of class 2

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(i) [M.Bacon, 1994] $\mathcal{N}_{n,2} \otimes \mathcal{N}_{n,2}$ is free abelian of rank $\frac{1}{3}n(n^2 + 3n - 1)$.
More precisely, we have
$\mathcal{N}_{n,2} \otimes \mathcal{N}_{n,2} \cong \Delta(F_n^{ab}) \times M(\mathcal{N}_{n,2}) \times \mathcal{N}'_{n,2}$.

(ii) [R. R., 2016] For $q > 1$ and q odd,
$\mathcal{N}_{n,2} \otimes^q \mathcal{N}_{n,2} \cong (C_q)^{((\binom{n+1}{2}))+M_n(3))} \times \mathcal{N}'_{n,2}\mathcal{N}^q_{n,2}$,
where $M_n(3) = \frac{1}{3}(n^3 - n)$ is the $q-$rank of $\gamma_3(\mathcal{N}_{n,2})/\gamma_3(\mathcal{N}_{n,2})^q\gamma_4(\mathcal{N}_{n,2})$.

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Consequently, for $q > 1$ and q odd,

$$d(\mathcal{N}_{n,2} \otimes^q \mathcal{N}_{n,2}) = \frac{1}{3}(n^3 + 3n^2 + 2n).$$

This is the least upper bound for $d(G \otimes^q G)$, G a class 2 nilpotent group with $d(G) = n$:

[E.Rodrigues, 2011]

Let G be a nilpotent group of class 2 with $d(G) = n$. Then $G \otimes^q G$ can be generated by at most $\frac{1}{3}n(n^2 + 3n + 2)$ elements. In particular, if G is finite and $\gcd(q, |G|) = 1$ then $d(G \otimes^q G) \leq n^2$.

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An example: The Heisenberg group in detail

Let $H = F_2/\gamma_3(F_2)$ be the Heisenberg group, where F_2 denotes the free group of rank 2.

By previous thm, part (ii) we have, for $q > 1$, q odd:

$H \otimes^q H \cong (C_q)^5 \times H'H^q$

Now, H has the polycyclic presentation

$$H = \langle x, y, z \mid \ [y, x] = z, [z, x] = 1 = [z, y]\rangle. \qquad (7)$$

and thus $\Upsilon^q(H)$ is generated by

$$\{[x, x^{\varphi}], [x, y^{\varphi}], [y, x^{\varphi}], [y, y^{\varphi}], [x, z^{\varphi}], [y, z^{\varphi}], \hat{x}, \hat{y}\}.$$

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In addition, the following relations hold in $\nu^q(H)$:

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and all other generators commute.

$\therefore\ \Upsilon^q(H)$ is a homomorphic image of the $\langle \hat{x}, \hat{y}, [y, x^{\varphi}], [x, z^{\varphi}], [y, z^{\varphi}], [x, y^{\varphi}][y, x^{\varphi}], [x, x^{\varphi}], [y, y^{\varphi}] \mid \text{above relations} \rangle$.

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Let G be polycyclic, given by a consistent polycyclic presentation, say $G = F/R$

Our procedure is an adaptation to all $q \geq 0$ of a method described by Eick and Nickel (2008) for the case $q = 0$

We can find a consistent polycyclic presentation for

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Let $\tau^q(G) := \nu^q(G)/\Delta^q(G)$. We have
$\tau^q(G) \cong (G \wedge^q G) \rtimes (G \times G)$.
Can find a consistent pcp for $\tau^q(G)$.
Need the concept of a $q-$biderivation, which extends the concept of a crossed pairing (biderivation) to the context of q-tensor squares
Crossed pairings have been used in order to determine homomorphic images of the non-abelian tensor square $G \otimes G$.
Let G and L be arbitrary groups.

Def. [I.Dias, 2011]

A function $\lambda : G \times G \times G \to L$ is called a *q-biderivation* if it satisfies the following properties, for all $g, g_1, h, h_1, k_1 \in G$:

$$(gg_1, h, k)\lambda = (g^{g_1}, h^{g_1}, 1)\lambda\,(g_1, h, k)\lambda$$
$$(g, hh_1, k)\lambda = (g, h_1, 1)\lambda\,(g^{h_1}, h^{h_1}, k)\lambda$$
$$((1,1,k)\lambda)^{-1}(g, h, 1)\lambda\,(1,1,k)\lambda = (g^{k^q}, h^{k^q}, 1)\lambda$$
$$(1,1,kk_1)\lambda = (1,1,k)\lambda \prod_{i=1}^{q-1}\left\{(k, (k_1^{-i})^{k^{q-1-i}}, 1)\lambda\right\}(1,1,k_1)\lambda$$
$$[(1,1,k)\lambda, (1,1,k_1)\lambda] = (k^q, k_1^q, 1)\lambda$$
$$(1,1,[g,h])\lambda = ((g,h,1)\lambda)^q.$$

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A consistent pcp for $\nu^q(G)$

Can compute the image of the action of G and of the $q-$biderivation λ in the polycyclic presentation of $G \wedge^q G \cong (G^*)'(G^*)^q$

This will give a consistent pcp for $\tau^q(G)$.

Now $\nu^q(G)$ can be polycyclicly presented as a certain central extension of $\tau^q(G)$.

Finally, standard methods for polycyclic groups are used to find a consistent pcp for the $q-$tensor square $G \otimes^q G$ as a subgroup of $\nu^q(G)$:

$$G \otimes^q G \cong [G, G^{\varphi}]\mathfrak{G}.$$

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Thank You!