Representations and subgroup structure of simple algebraic groups

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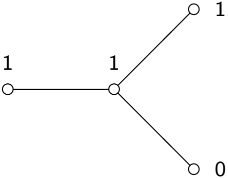
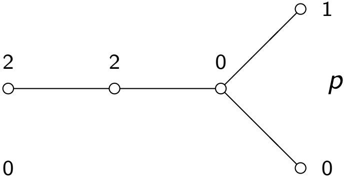
EPF Lausanne

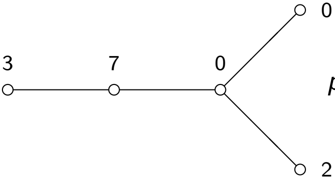
Groups St Andrews 2017



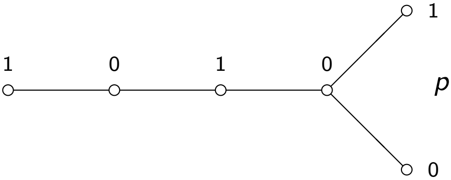
$p = 3$



$p = 5$



$p = 11$



$p = 2$

Irreducible triples (H, K, V)

Table: Irreducible triples

$H < K$	$V\|_H$	$V\|_K$	conditions
$C_n < A_{2n-1},\ n \geq 2$	$a\omega_1$	$a\lambda_1$	$a \geq 2$
$C_n < A_{2n-1},\ n \geq 2$	$a\omega_j + b\omega_{j+1}$, $j < n$	$a\lambda_j + b\lambda_{j+1}$	$a + b = p - 1 > 1$ $a \neq 0$ if $j = n - 1$

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$B_n < A_{2n},\ n \geq 3$	$\omega_j,\ 2 \leq j < n$	λ_j	$p \neq 2$
$B_n < A_{2n},\ n \geq 2$	$2\omega_n$	λ_n	$p \neq 2$

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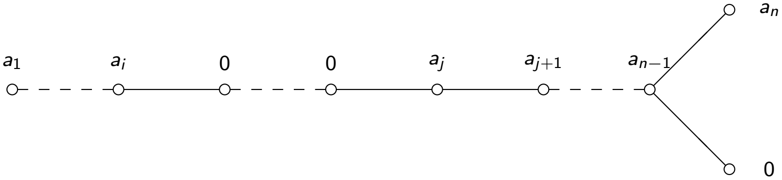
$D_n < A_{2n-1},\ n \geq 4$	$\omega_j,\ 2 \leq j < n - 1$	λ_j	$p \neq 2$
$D_n < A_{2n-1},\ n \geq 4$	$\omega_{n-1} + \omega_n$	λ_{n-1}	$p \neq 2$
$\vdots$	$\vdots$	$\vdots$	

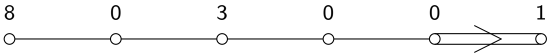
$B_n \subset D_{n+1}$



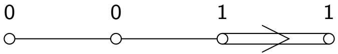
$$a_n \neq 0,\ a_i + a_j + j - i \equiv 0 \mod p,$$

$$\text{whenever } a_i a_j \neq 0 \text{ and } a_i\ 0 \cdots\ 0\ a_j$$

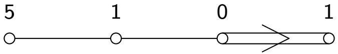
Seitz 1987, Cavallin-T 2017



$p = 13$



$p = 5$



$p = 7$

Positive-dimensional closed irreducible subgroups, set-up

Problem: classify all triples (H, K, V):

K, a simply connected simple algebraic group,

V, a non-trivial irreducible K-module, p-restricted, tensor-indecomposable, not the natural module or its dual if K is classical,

H, closed positive-dimensional subgroup of K, $HZ(K)/Z(K)$ disconnected, acting irreducibly on V, with $V|_{H^\circ}$ reducible.

Positive-dimensional closed irreducible subgroups, results

Complete answer in the following cases:

Ford (1996, 1999). K classical, H° simple such that $V|_{H^\circ}$ has p-restricted summands.

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Burness, Ghandour, Marion, T. (2015) K classical, H maximal non geometric subgroup of K.

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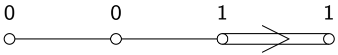
Burness, Marion, T. (2017). K classical, H non maximal in K, and if H is a *decomposition subgroup*† of an orthogonal type group K, then V is not a spin module.

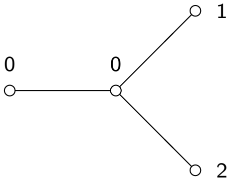
† H permutes the summands of an orthogonal decomposition $W = W_1 \perp \cdots \perp W_t$, where $K = \mathrm{SO}(W)$.

Example: $D_4.2 \subset B_4$

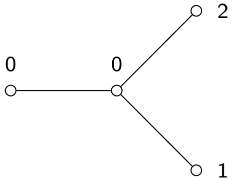


dimension 336

$p = 5$



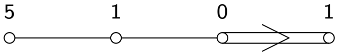
$\oplus$



dimension 168

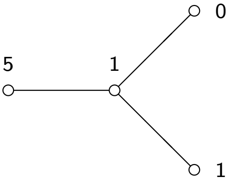
dimension 168

Another example

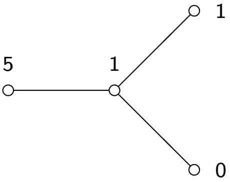


$p = 7$

dimension ???



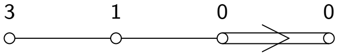
$\oplus$



dimension ???

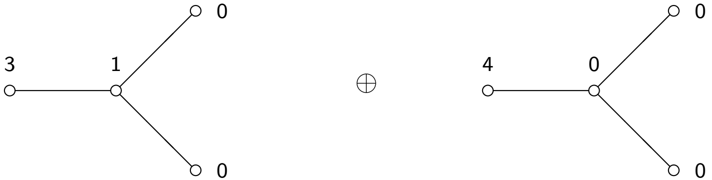
dimension ???

Non-irreducible action, branching rule



dimension 957

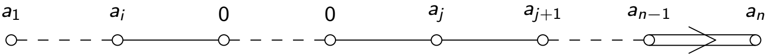
$p = 5$



dimension 664

dimension 293

$D_n \subset B_n$, exactly two summands



- $a_n \in \{0, 1\}$,
- $2(a_t + a_n) + 2(n - t) - 1 \equiv 0 \mod p$, $t < n$ maximal such that $a_t \neq 0$,
- $a_i + a_j + j - i \equiv 0 \mod p$, whenever $1 \leq i < j \leq t$, $a_i a_j \neq 0$ and $a_r = 0$ for all $i < r < j$.

If $a_n = 1$, $D_n.2$ acts irreducibly.

Ford 1996, Cavallin 2015

Irreducible chains, connected

Corollary (Seitz, Cavallin-T.)

Let $\rho : H \to \mathrm{SL}(V)$ be an irreducible, tensor indecomposable representation of a simple algebraic group H, and let G be the smallest classical group containing $\rho(H)$. If K is a closed connected subgroup of G with $\rho(H) < K < G$ (proper containments), then with precisely two exceptions, K is maximal among closed connected subgroups of G. The exceptions are as follows:

1. $p = 3$, $A_2 < G_2 < B_3 < \mathrm{SO}_{27}$;
2. $p = 2$, $D_4 < C_4 < F_4 < \mathrm{SO}_{26}$.

Irreducible chains, general

G, a simply connected cover of a simple classical algebraic group.

V, a nontrivial p-restricted irreducible tensor-indecomposable G-module such that:

$V \neq W^\tau$ for any automorphism τ of G, where W is the natural G-module.

Write $\ell = \ell(G, V)$ for the length of the longest chain of closed positive-dimensional subgroups

$$H_\ell < H_{\ell-1} < \cdots < H_2 < H_1 = G$$

such that $V|_{H_\ell}$ is irreducible.

Corollary (Burness-Marion-T.)

Let G and V be as above, and assume V is not a spin module. Then either $\ell(G, V) \leqslant 5$, *or*
$G = \mathrm{SL}(W)$ *and* $V \in \{\wedge^2(W), \wedge^3(W), \wedge^2(W)^*, \wedge^3(W)^*\}$.

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Remark

If G is an orthogonal group (or a symplectic group with $p = 2$) and V is a spin module, then $\ell(G, V)$ can be arbitrarily large; the same is true if $V = W$ or W^* and for the list of exceptions for $\mathrm{SL}(W)$.