Some finiteness conditions on centralizers or normalizers in groups

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"Groups St Andrews in Birmingham - 2017"
August 2017

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If G is a torsion free BCI-group, then G is abelian.

A non abelian free group is an FCI-group which is not a BCI-group!

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Let G be an FCI-(BCI-)group and $N \triangleleft G$, N finite.
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Proposition

Let G be an FCI-(BCI-)group and $N \triangleleft G$, N finite.
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Let G be periodic, $x \in G$, $|x| = 2$, $|C_G(x)| < \infty$.

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Theorem [Meixner, Hartley, Pattet, Khukhro]

Let G be locally finite, $x \in G$, $|x| = p$, $|C_G(x)| < \infty$.
Then, G is nilpotent-by-finite.

Shalev (1994)

G satisfies (*) iff $|C_G(x)|$ finite or $|G : C_G(x)|$ finite, for all $x \in G$.

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G locally finite, FCI-group $\Longrightarrow |G : FC(G)|$ finite

Theorem [Fernández-Alcober, Legarreta, Tortora, T.]

Let $D = Q \times A$ an infinite, periodic, Dedekind group, where $Q \cong 1$ or $Q \cong Q_8$.

G is an infinite, locally finite, FCI-group iff

$G = D$, or $G = \langle D, x \rangle$, D of finite 2-rank, x acts on D as a power automorphism and there exists $m > 1$ such that $x^m \in D$ and $|C_A(x^k)|$ is finite, $\forall k = 1, \ldots, m-1$.

G. A. Fernández-Alcober, L. Legarreta, A. Tortora, and M. Tota, *A finiteness condition on centralizers in locally finite groups*, Monatsh. Math. **183** (2017), no. 2, 241–250.

Corollary

Let G be a locally finite group. Then the following facts are equivalent:

1. G is an FCI-group
2. G is a BCI-group

Recall that a group is *locally graded* if every non-trivial finitely generated subgroup has a non-trivial finite image.

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Tarski monster groups are periodic BCI-groups, which are not locally graded.

Theorem [Fernández-Alcober, Legarreta, Tortora, T.]

Every locally graded periodic BCI-group is locally finite.

Questions

Does the previous theorem hold for FCI-groups?

Given a periodic residually finite group G in which the centralizer of each non-trivial element is finite, is G finite?

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Given a periodic residually finite group G in which the centralizer of each non-trivial element is finite, is G finite?

Examples

There exist finitely generated infinite periodic groups which are residually finite but not FCI (Golod, Grigorchuk and Gupta-Sidki).

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G. A. Fernández-Alcober, L. Legarreta, A. Tortora, and M. Tota, *Some finiteness conditions on normalizers or centralizers in groups*, Comm. Algebra, in press.

Theorem [Fernández-Alcober, Legarreta, Tortora, T.]

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Corollary

Let G be locally graded and periodic. Then the following facts are equivalent:

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2. G is a BCI-group

Theorem (periodic case) [Fernández, Legarreta, Tortora, T.]

Let G be a non-Dedekind infinite periodic group. Then G is a locally nilpotent FCI-group if and only if $G = P \times Q$, where P and Q are as follows:

1. $P = \langle g, A \rangle$ is a 2-group, where A is infinite abelian of finite rank, and g is an element of order at most 4 such that $g^2 \in A$ and $a^g = a^{-1}$ for all $a \in A$.
2. Q is a finite abelian $2'$-group.

G. A. Fernández-Alcober, L. Legarreta, A. Tortora, and M. Tota, *A finiteness condition on centralizers in locally nilpotent groups*, Monatsh. Math. **182** (2017), no. 2, 289–298.

Theorem (non periodic case) [Fernández, Legarreta, Tortora, T.]

Let G be a non-periodic group. Then G is a locally nilpotent FCI-group if and only if either G is abelian, or $G = \langle x \rangle \ltimes D$ where

- x is of infinite order and acts on D as a power automorphism,
- $D = Q \times A$ is a Dedekind group, direct product of finitely many p-groups of finite rank, and
- $C_A(x^k)$ is finite for every $k \geq 1$.

Theorem [Fernández-Alcober, Legarreta, Tortora, T.]

Let G be a non-periodic group. Then the following conditions are equivalent:

1. G is a BCI-group.
2. There exists $n \in \mathbb{N}$ such that $|C_G(x)| \leq n$ whenever $\langle x \rangle \ntrianglelefteq G$.
3. Either G is abelian or $G = \langle g, A \rangle$, where A is a non-periodic abelian group of finite 2-rank and g is an element of order at most 4 such that $g^2 \in A$ and $a^g = a^{-1}$ for all $a \in A$.

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Corollary

Let G be a non-periodic locally nilpotent group.
If G is a BCI-group then G is abelian.

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Corollary

Let G be a non-periodic locally nilpotent group.
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There exist non-periodic locally nilpotent FCI-groups, which are not BCI-groups!

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Theorem [Fernández-Alcober, Legarreta, Tortora, T.]

Let G be a non-periodic group. Then the following hold:

G is a BNI-group if and only if either G is abelian or $G = \langle g, A \rangle$, where A is a non-periodic abelian group of finite 0-rank and finite 2-rank, and g is an element of order at most 4 such that $g^2 \in A$ and $a^g = a^{-1}$ for all $a \in A$.

Robinson (2016)

FCI-groups and FNI-groups have been classified in the locally (soluble-by-finite) case.

D. J. S. Robinson, *On groups with extreme centralizers and normalizers*, Adv. Group Theory Appl. **1** (2016), 97–112.

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Set $D := FC(G) \Rightarrow D$ finite $\Rightarrow G/D$ infinite locally graded BCI.

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It follows $|C_G(x) : \langle x \rangle| \leq n$ for all $x \in G \setminus \{1\}$.
So, each finite quotient of G is soluble and has not elements of order pq.

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G/R f. g., res. fin., $\exp(G/R) < \infty \Rightarrow G/R$ finite $\Rightarrow 1 \neq R$ f. g.

Then, $\exists\ K < R\ :\ |R : K| < \infty \Rightarrow |G : K| < \infty \Rightarrow R \leq K$.

Contradiction!

Thank you!