Constructing Majorana representations

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Joint work with M. Pfeiffer, St Andrews

The Monster group

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- If $t, s \in 2A$ then ts is of order at most 6 and belongs to one of nine conjugacy classes:

$$1A, 2A, 2B, 3A, 3C, 4A, 4B, 5A, 6A.$$

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- The 2A-axes generate the Griess algebra i.e. $V_{\mathbb{M}} = \langle\langle \psi(t) \,:\, t \in 2A \rangle\rangle$
- If $t, s \in 2A$ then the algebra $\langle\langle \psi(t), \psi(s) \rangle\rangle$ is called a dihedral subalgebra of $V_{\mathbb{M}}$ and has one of nine isomorphism types, depending on the conjugacy class of ts.

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The algebra V also contains the axis $\psi(ts)$. In fact, it is of dimension 3:

$$V = \langle \psi(t), \psi(s), \psi(ts) \rangle_{\mathbb{R}}.$$

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- In particular, we have $\mathrm{Aut}(V^\#) = \mathbb{M}$ and $V_2^\# \cong V_{\mathbb{M}}$.

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then the tuple (G, V, T) is called a Majorana representation.

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Sakuma's Theorem (A. A. Ivanov et al, 2010)

Any Majorana algebra generated by two Majorana axes is isomorphic to a dihedral subalgebra of the Griess algebra.

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He never published his code or the full details of his algorithm and reproducing his work has been an important aim of the theory ever since.

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If at any point in the algorithm a contradiction with the Majorana axioms is found, an appropriate error message is returned.

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- $A_8, L_2(11), M_{11}$ and beyond!